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Oscillation Criteria of a Class of Third Order Nonlinear Difference Equations

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Abstract

In this paper, a class of third order non-linear difference equations with deviating argument, which is of the form

$$\Delta(a_n(\Delta b_n(\Delta x_{n)})^{\alpha})^{\beta}) + c_n x^{\mu}_{n+\tau} = 0$$

is considered. Sufficient conditions for oscillation and almost oscillation are obtained. Examples are provided to interpret the results.

Keywords: *Difference Equations, Oscillation, Almost Oscillation, Quickly Oscillation.*

Mathematics Subject Classification: 39A10, 39A21.

1. Introduction

In past one decade, there has been a lot of activity done on the oscillatory theory for second order and fourth order nonlinear difference equations [1], [2], [7], [8], [10], [11]. In the survey of literature the attention given on third order difference equation is less with the second and fourth order difference equations.

This paper, we investigate the oscillation of a class of third order nonlinear difference equations of the form

$$\Delta(a_n(\Delta b_n(\Delta x_n)^{\alpha})^{\beta}) + c_n x^{\mu}_{n+\tau} = 0 \qquad (1.1)$$

where α, β, μ are the ratios of odd positive integers, $\tau \in Z$, is a deviating argument, $\{a_{n\}}, \{b_n\}, and\ \{c_n\}$ are positive real sequences defined for $n \in N_o = \{n_o + n_{o+1} + \cdots.\}, n_o$ is a positive integer. The forward difference operator, Δ is defined by $\Delta x_n = x_{n+1} - x_{n.}$

By a solution of (1.1), we mean a real sequence $\{x_n\}$ that satisfies (1.1) for all $n \in N_o$ .A nontrivial solution $\{x_n\}$, $n \in N_o$ of (1.1) is oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is non-oscillatory. Equation (1.1) is said to be oscillatory, if all its solutions are oscillatory.

A solution {x_n} of (1.1) is quickly oscillatory if

$$x_n = (-1)^n o_{n,} \quad o_n > 0, \ for \quad n > 0$$

Equation (1.1) is almost oscillatory if either $\{x_n\}$ is oscillatory or Δx_n is oscillatory or $\lim_{n\to\infty} x_n = 0.$

The motivation behind the work is mainly derived from the oscillatory solutions of non-linear difference equations contained in [1], [8] and the knowledge gained from [2 – 7], [9], [10]. oscillatory and asymptotic properties of third order difference equations are established.

Consider (1.1) as a three dimensional system, let

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Consider (1.1) as a three dimensional system, let

$$y_n = b_n(\Delta x_n)^{\alpha},\ z_n = a_n(\Delta y_n)^{\beta}. \qquad (1.3)$$

Construct the nonlinear system,

$$\begin{Bmatrix} \Delta x_n = B_n y_n^{1/\alpha}, \\ \Delta y_n = A_n z_n^{1/\beta}, \\ \Delta z_n = -C_n x_{n+\tau}^{\mu} \end{Bmatrix} \qquad (1.4)$$

where $B_n = b_n^{-1/\alpha}, A_n = a_n^{-1/\beta} \,\&\, C_n = c_n$. If any one solution of (1.4) is positive then other two solutions also positive. If any one solution of (1.4) is negative then other two solutions also negative.

The canonical form of the difference operator in (1.1) is defined by,

$$\sum_{n=n_0}^{\infty} a_n^{-1/\alpha} = \sum_{n=n_0}^{\infty} b_n^{-1/\beta} = \infty. \qquad (1.5)$$

In section 2, sufficient conditions are obtained for quickly oscillatory solutions of (1.1). In section 3, some sufficient conditions for oscillatory and non-oscillatory solutions of (1.1) are presented. Section 4, deals with the almost oscillatory solutions. In section 5, examples are given to illustrate the results.

2. Quickly Oscillatory Solutions

Theorem 2.1: Assume that and is even. If is odd, then (1.1) has no quickly oscillatory solutions.

Proof: Let $x_n = (-1)^n a_n$ be a quickly oscillatory solution (1.1) with positive even terms.

Then there exists $n \in N_0.0_n > 0$ such that $\Delta x_n = (-1)^{n+1}(o_{n+1} + o_n)$.

From the first equation of (1.4), we have

$$y_n = \left(\frac{\Delta x_n}{B_n}\right)^{\alpha} = (-1)^{n+1}\left(\frac{o_{n+1} + o_n}{B_n}\right)^{\alpha} = (-1)^{n+1} p_n$$

where $p_n = \left[\frac{o_{n+1}}{B_n} + \frac{o_n}{B_n}\right]^{\alpha} > 0.$

From second equation of (1.4),

$$z_n = \left(\frac{\Delta y_n}{A_n}\right)^{\beta} = (-1)^n\left[\frac{p_{n+1} + p_n}{A_n}\right]^{\beta} = (-1)^n q_n.$$

where $q_n = \left[\frac{p_{n+1}}{A_n} + \frac{p_n}{A_n}\right]^{\beta} > 0.$ Repeating this process,

$$\Delta z_n = (-1)^{n+1}(q_{n+1} + q_n) = -C_n(-1)^{(n+\tau)\mu} o_{n+\tau}^{\mu}$$

$$= C_n(-1)^{(n+1+\tau)\mu} o_{n+\tau}^{\mu}.$$

Since τ is odd, therefore (1.1) has quickly oscillatory solution with positive odd terms, which gives contradiction.

Remark 2.2: If τ is even, $n > 0$, and n is odd, then (1.1) has no quickly oscillatory solution.

Theorem 2.3: Let α, β and μ be the ratios of odd positive sequences. If $\tau = 0,$ then (1.1) has quickly oscillatory solutions.

Proof Let x_n be not quickly oscillatory solution of (1.1),

That means $x_n = (-1)^n o_n, o_n < 0$

Since $\tau = 0$, α, β and μ are ratios of odd positive integers. Let assume that $\alpha = \beta = \mu = 1.$

Since $x_n = (-1)^n o_n\ o_n < 0,$ then it follows

$$\Delta x_n = (-1)^n[o_{n+1} + o_n].$$

$$\therefore \Delta(b_n \Delta x_n) = (-1)^{n+2}\left[b_{n+1}o_{n+2} + (b_{n+1} + b_n)o_{n+1} + b_n o_n\right]$$

$$\Delta(c_n\Delta(b_n\Delta(x_n))) = (-1)^{n+1}\left[b_{n+2}o_{n+3} + (b_{n+2}c_{n+1} + b_{n+1}c_{n+1} + b_{n+1}c_n)o_{n+2} + b_{n+1}c_{n+1} + b_{n+1}c_n + b_n c_n)o_{n+1} + b_n c_n o_n\right]$$

(2.1)

Taking $\alpha = \beta = \mu = 1$ and $\tau = 0$ in (2.1), then it becomes

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$$\Delta(c_n\Delta(b_n\Delta(x_n))) = -c_n x_n \qquad (2.2)$$

Comparing (2.1) and (2.2), we get

$$(-1)^{n+1}\left[b_{n+2}o_{n+3} + (b_{n+2}c_{n+1} + b_{n+1}c_{n+1} + b_{n+1}c_n)o_{n+2} + (b_{n+1}c_{n+1} + b_{n+1}c_n + b_n c_n)o_{n+1} + b_n c_n o_n\right] = -c_n x_n$$
(2.3)

According to the assumption $o_n < 0,$ then the left side terms of (2.3) are positive. But the right side terms of (2.3) are negative, which is a contradiction.

Hence the solutions of x_n of (1.1) are quickly oscillatory solutions.

3. Oscillatory Solutions

Lemma 3.1: The followings are equivalent.

(i) x is a solution of (1.1).

(ii) $y = y_n$, where $y_n = b_n(\Delta x_n)^{\alpha}$, is a solution of

$$\Delta\left(\frac{1}{c_n^{1/\mu}}\left(\Delta\left(a_n(\Delta y_n)^{\beta}\right)\right)^{1/\mu}\right) + \frac{1}{b_{n+\tau}^{1/\alpha}} y_{n+\tau}^{1/\alpha} = 0 \qquad (3.1)$$

(iii) $z = z_n$, where $z_n = a_n(\Delta y_n)^{\beta}$, is a solution of

$$\Delta\left(b_{n+\tau}\left(\Delta\frac{1}{c_n^{1/\mu}}(\Delta z_n)^{1/\mu}\right)\right) + \frac{1}{a_{n+\tau}^{1/\beta}} z_{n+\tau}^{1/\beta} = 0. \qquad (3.2)$$

Proof: Let us prove (i) is equivalent to (ii).

Consider the third equation in (1.4) and (1.1), we express as follows,

$$x_{n+\tau} = -\frac{1}{c_n^{1/\mu}}(\Delta z_n)^{1/\mu} = -\frac{1}{c_n^{1/\mu}}\left(\Delta\left(a_n(\Delta y_n)^{\beta}\right)\right)^{1/\mu}$$
(3.3)

Next, we consider the first equation in (1.4), we have

$$\Delta x_{n+\tau} = -\Delta\left(\frac{1}{c_n^{1/\mu}}\left(\Delta\left(a_n(\Delta y_n)^{\beta}\right)\right)^{1/\mu}\right) = \frac{1}{b_{n+\tau}^{1/\alpha}} y_{n+\tau}^{1/\alpha}$$

$$\Delta\left(\frac{1}{c_n^{1/\mu}}\left(\Delta(a_n(\Delta y_n)^{\beta})\right)^{1/\mu}\right) + \frac{1}{b_{n+\tau}^{1/\alpha}} y_{n+\tau}^{1/\alpha} = 0$$

which gives (ii).

Next, to prove (i) is equivalent to (iii). From (3.3),

$$\Delta x_n = -\Delta\left(\frac{1}{c_{n-\tau}^{1/\mu}}\left(\Delta(a_{n-\tau}(\Delta y_{n-\tau})^{\beta})\right)^{1/\mu}\right).$$

Substitute this into $\Delta y_n = \Delta(b_n(\Delta x_n)^{\alpha}),$ we get

$$\Delta y_n = \Delta\left(b_n\left(-\Delta\left(\frac{1}{c_{n-\tau}^{1/\mu}}\left(\Delta a_{n-\tau}(\Delta y_{n-\tau})^{\beta}\right)^{1/\mu}\right)\right)^{\alpha}\right)$$

From second equation in (1.4), we get

$$\Delta y_{n+\tau} = \Delta\left(b_{n+\tau}\left(-\Delta\left(\frac{1}{c_n^{1/\mu}}\Delta\left(a_n(\Delta y_n)^{\beta}\right)\right)^{1/\mu}\right)\right)^{\alpha}$$

$$= -\Delta\left(b_{n+\tau}\left(\Delta\left(\frac{1}{c_n^{1/\mu}}\left((\Delta z_n)^{1/\mu}\right)^{1/\mu}\right)\right)\right) = \frac{1}{a_{n+\tau}^{1/\beta}} z_{n+\tau}^{1/\beta}$$

which gives (iii).

Theorem 3.2: (1.1) is oscillatory $\Leftrightarrow$ (3.1) & (3.2) are oscillatory.

Proof: Equation (1.1) is oscillatory.

$\Leftrightarrow$ Every solution (1.1) is oscillatory.

$\Leftrightarrow$ x_n is an oscillatory solution of (1.1) for $n \in N_0$.

$\Leftrightarrow$ y_n is an oscillatory solution of (3.1) for $n \in N_0$.

Lemma 3.3: Assume (1.5), then any solution of (x, y, z) of (1.4) so that $x_n > 0$ for the large value of n, is of the following types:

(B_1) $x_n > 0, y_n > 0, z_n > 0$ for all the large value of n.

(B_2) $x_n > 0, y_n < 0, z_n > 0$ for all the large value of n.

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Proof: Let (x, y, z) be non-oscillatory solution of (1.4).

Therefore, there exists a solution such that $y_n > 0, z_n < 0$ for the large value of n.

Since $\Delta z_n < 0,$ there exists $k > 0$ such that $z_n \le -k,$ for the large value of n. Summing the second equation of (1.4),

$$y_n - y_{n_0} = \sum_{i=n_0}^{n-1} A_i z_i^{1/\beta} \le z_n^{1/\beta} \sum_{i=n_0}^{n-1} A_i \le -k^{1/\beta} \sum_{i=n_0}^{n-1} A_i .$$

Taking $n \to \infty,$ which implies

$$\lim_{n\to\infty} y_n = -\infty$$

Similarly, the result is true for a solution $y_n < 0, z_n < 0,$ for the large value of n. Summing the first equation in (1.4), then

$$x_n - x_{n_0} = \sum_{i=n_0}^{n-1} B_i y_i^{1/\alpha} \le y_n^{1/\alpha} \sum_{i=n_0}^{n-1} B_i \le -k^{1/\alpha} \sum_{i=n_0}^{n-1} B_i .$$

Taking $n \to \infty$, & we get , $\lim_{n\to\infty} x_n = -\infty$ which is a contradiction.

Lemma 3.4: Equation (1.1) has solution of type (B_1), if the following are not hold

$$(i) \sum_{n=n_0}^{\infty} c_n \left(\sum_{i=n_0}^{n+r-1} \frac{1}{b_i^{1/\alpha}} \right)^{\mu} = \infty.$$

(3.4)

$$(ii) \sum_{n=n_0}^{\infty} c_n \left(\sum_{i=n_0}^{n+r-1} \frac{1}{b_i^{1/\alpha}} \left(\sum_{j=n_0}^{i-1} \frac{1}{a_j^{1/\beta}} \right)^{\alpha} \right)^{1/\mu} = \infty.$$

(3.5)

Proof: Let (the solution of (1.4)) (x, y, z) be a solution of type (B_1), that is all the solutions are positive.

There exists $k > 0$ & z is positive increasing such that $z_n^{1/\beta} \ge k$ for large $n, n \ge n_0$.

From the first and second equations in (1.4), we get

$$x_i = \sum_{i=n_0}^{j-1} B_i y_i^{1/\alpha}$$

$$y_i = \sum_{i=n_0}^{j-1} A_i z_i^{1/\beta} \ge z_n^{1/\beta} \sum_{i=n_0}^{j-1} A_i \ge k \sum_{i=n_0}^{j-1} A_i$$

$$x_i \ge y_n^{1/\alpha} \sum_{n=n_0}^{j-1} B_n \ge k^{1/\alpha} \sum_{n=n_0}^{j-1} B_n \left(\sum_{k=n_0}^{n-1} A_i \right)^{1/\alpha} \qquad (3.6)$$

Let us assume (3.4) and (3.5) hold. By assuming third equation of (1.4) and using (3.6), we get

$$z_{n_0} - z_n = -\sum_{i=n_0}^{n-1} \Delta z_i \ge \sum C_i x_{n+\tau}^{\mu}$$

$$z_{n_0} - z_n \ge k^{\mu/\alpha} \sum_{i=n_0}^{n-1} C_i \left(\sum_{j=n_0}^{i+r-1} B_j \left(\sum_{k=n_0}^{j-1} A_k \right)^{1/\alpha} \right)^{\mu}$$

Therefore (1.1) has no solution of type (B_1).

This completes the proof.

Lemma 3.5:

Let $\sum_{n=n_0}^{\infty} c_n < \infty$ be hold.

Then (1.1) has no solution of type (B_2) if any one of the following conditions hold, (i)

$$T := \sum_{n=n_0}^{\infty} \frac{1}{a_n} \left(\sum_{k=n}^{\infty} c_k \right)^{\frac{1}{\beta}} = \infty$$

(3.7)

(ii) $T < \infty$

$$\sum_{n=n_0}^{\infty}\frac{1}{b_n^{1/\alpha}}\left(\sum_{n=n_0}^{\infty}\frac{1}{a_n^{1/\beta}}\left(\sum_{n=n_0}^{\infty}c_i\right)^{1/\beta}\right)^{1/\alpha}=\infty$$

(3.8)

Proof: Let (z, a, b) be a solution of (1.4) and satisfying $z_n > 0, a_n < 0, b_n > 0.$

Since the solutions z and $-y$ are positive and decreasing, we get

$$\lim_{n\to\infty} z_n = z_\infty,\ z_\infty \geq 0$$

$$\lim_{n\to\infty} y_n = y_\infty,\quad y_\infty \leq 0$$

By summation of the third equation of (1.4), we get

$$z_n - z_{n_0} = -\sum_{n=n_0}^{n-1} C_k x_{k+\tau}^{\mu}$$

which implies,

$$z_n = z_\infty + \sum_{k=n_0}^{\infty} C_k x_{k+\tau}^{\mu} \geq x_{k+\tau}^{\mu}\sum_{k=n_0}^{\infty} C_k.$$

Let us assume (i) hold, then summing second equation of (1.4), we get

$$y_m - y_{n_0} = \sum_{n=n_0}^{m-1} A_n z_n^{1/\beta} \geq x_{k+\tau}^{\mu/\beta}\sum_{n=n_0}^{m-1} A_n\left(\sum_{k=n}^{\infty} C_k\right)^{1/\beta}$$

$$y_m \geq y_{n_0} + x_{k+\tau}^{\mu}\sum_{n=n_0}^{\infty} A_n\left(\sum_{k=n}^{\infty} C_k\right)^{1/\beta}$$

which is a contradiction.

Let (ii) hold, consider the second equation of (1.4), we have

$$-y_n = -y_\infty + \sum_{k=n}^{\infty} A_k z_k^{1/\beta}$$

$$-y_n \geq x_{n+\tau}^{\mu/\beta}\sum_{k=n}^{\infty} A_k\left(\sum_{j=n}^{\infty} C_j\right)^{1/\beta} \qquad (3.9)$$

Since x is positive decreasing and using (4.1), we have

$$x_{n_0} = x_n + \sum_{n=n_0}^{n-1} B_k(-y_k)^{1/\alpha}$$

$$\geq \sum_{k=n}^{\infty} B_k\left(\sum_{i=n}^{\infty} A_i\left(\sum_{j=n}^{\infty} C_j\right)^{1/\beta}\right)^{1/\alpha}$$

which is a contradiction.

Therefore (1.1) has no solution of type (B_2).

This completes the proof.

Theorem 3.6: Assume that (1.5), $\sum_{n=n_0}^{\infty} c_n < \infty$ and $\tau \in Z,$ if (3.5) and (3.8) hold then (1.1) is oscillatory.

Proof From the lemma 3.4 and lemma 3.5, if the conditions (3.5) and (3.7) hold, then (1.1) has no solutions of type (B_1) and (B_2).

By lemma 3.3, (1.1) has oscillatory solutions.

Theorem 3.7: Assume that (1.5), $\sum_{n=n_0}^{\infty} c_n < \infty$ and $\tau \in Z,$ if (3.6) and (3.9) hold then (1.1) is oscillatory.

Proof From the lemma 3.4 and lemma 3.5, (1.1) has no solutions of type (B_1) and (B_2) if the conditions (3.8) hold.

Then by lemma (3.3), (1.1) is oscillatory.

4. Almost Oscillatory Solutions

Throughout this section, the conditions of almost oscillatory solutions of (1.1) are obtained.

Corollary 4.1: If x, y, z is a solution of (1.4), with bounded first component and such that one of its components is of one sign, then there exists limit of

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sequence (x_n) and exactly one of the following two cases are hold

(i) $\lim_{n\to\infty} x_n \neq 0$ and sequence x, y and z are monotonic for the large value of n, or

(ii) sequence (y_n) is of one sign and $\lim_{n\to\infty} x_n = 0$.

Corollary 4.2: Assume

$$\sum_{n=n_0}^{\infty} A_n = \sum_{n=n_0}^{\infty} B_n = \infty$$

and x, y, z is a solution of (1.4), so that $\lim_{n\to\infty} x_n \in R,$ then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0.$$

Proof Since $\lim_{n\to\infty} x_n$ is finite. Consider the first equation in system (1.4),

$$\Delta x_n = B_n y_n^{1/\alpha}$$

Summing the equation, we get

$$x_n = x_{n_0} + \sum_{i=n_0}^{\infty} B_i y_i^{1/\alpha}$$

Now, assume the contrary that $\lim_{n\to\infty} y_n > 0$

$$\therefore \lim_{n\to\infty} y_n^{1/\alpha} > 0$$

Since B_n is positive and taking n tends to infinity, we have $\lim_{n\to\infty} x_n = 0$ which is a contradiction to the fact that $\lim_{n\to\infty} x_n$ is finite.

Hence $\lim_{n\to\infty} y_n = 0.$

Similarly we can prove $\lim_{n\to\infty} z_n = 0.$

Theorem 4.3: If $\lim_{n\to\infty} x_n \in R$ and $\sum_{i=1}^{\infty} c_i$ is divergent, then all the solutions of (1.1) is almost oscillatory.

Proof: Assume the contrary that (1.1) has a non – oscillatory solution does not approach zero. Now, we assume that $x_n > 0$ for the large value of n.

From corollary 4.1, $\{x_n\}$ exists. We have $\lim_{n\to\infty} x_n = k \in (0,\infty).$

Since $x_{n+r} > 0,$ there exists a positive integer n_1 such that

$$x_{n+r} \geq \frac{k^{\mu}}{2} \text{ for } n \geq n_1$$

Since $\{c_n\}$ is a real positive sequence. Summing the equation (4.1), we get

$$\sum_{i=n_1}^{\infty} c_i x^{\mu}{}_{i+r} \geq \frac{k^{\mu}}{2} \sum_{i=n_1}^{\infty} c_i = \infty$$

Summing the third equation of (1.4), we have

$$z_n - z_1 = -\sum_{i=1}^{n-1} c_i x^{\mu}{}_{i+r}$$

By corollary 4.2, we have $\lim_{n\to\infty} z_n = 0.$

Taking n tends to infinity in above equation, we get

$$z_1 = \sum_{i=1}^{\infty} c_i x_{i+r}^{\mu}$$

which gives a contradiction to the fact that z_1 is a constant term. Therefore, any bounded solution of (1.1) is almost oscillatory.

5. Examples

Example 5.1: Consider the third order difference equation

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$$\Delta\left(2^n\left(\Delta 4^n\left(\Delta x_n\right)\right)\right)+1300\left(2^{3n}\right)x_n=0. \tag{5.1}$$

Here $a_n=2n, b_n=4^n$, $c_n=1300\left(2^{3n}\right)$ and $\alpha=\beta=\mu=1.$ $x_n=(-1)^n 3^n$ is one of the quickly oscillatory solution of (5.1).

Example 5.2: Consider the difference equation of order 3,

$$\Delta((n-1)(\Delta^2 x_n))+\frac{1}{n-1}x_{n+3}^{\mu}=0,(\mu\geq 1). \tag{5.2}$$

Here $a_n=n-1, b_n=1$ and $c_n=\frac{1}{n-1}$ and $\alpha=\beta=1.$ we have

$$\sum_{n=n_0}^{\infty}(n-1)^{-1}=\sum_{n=n_0}^{\infty}1=\infty,$$

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{n-1}\right)\left(\sum_{n=n_0}^{\infty}\frac{1}{n-1}\right)=\infty$$

and $$\left(\sum_{n=n_0}^{n-1}1\right)\left(\sum_{n=n_0}^{n-1}\frac{1}{n-1}\right)\left(\sum_{n=n_0}^{n-1}\frac{1}{n-1}\right)=\infty$$

Therefore if $\mu>1,$ then the conditions (3.5) and (3.8) are satisfied and by theorem 3.6, (5.2) has no solution of type (B_2), therefore (5.2) is oscillatory.

Example 5.3: Suppose that $a_n=\frac{1}{n}, b_n=\frac{1}{n-1}$ and $c_n=2n.$ Let $\alpha=\beta=1.$ Take the deviating argument τ is 2 then the equation (1.1) becomes

$$\Delta\left(\frac{1}{n}\left(\Delta\frac{1}{n-1}(\Delta x_n)\right)\right)+2nx_{n+2}^{\mu}=0 \quad (\mu\geq 1) \tag{5.3}$$

Thus,

$$\sum_{n=n_0}^{\infty}2n\left(\sum_{n=n_0}^{n+1}(n-1)\left(\sum_{n=n_0}^{n}n\right)\right)^{1/\mu}=\infty$$

and

$$\sum_{n=n_0}^{n+1}(n-1)\left(\sum_{n=n_0}^{n+1}n\left(\sum_{n=n_0}^{n+1}2n\right)\right)=\infty.$$

Therefore if $\mu\geq 1,$ the conditions (3.6) and (3.9) are satisfied. Hence by theorem 3.7, (5.3) has no solution of type (B_2) , hence (5.3) is oscillatory.

Example 5.4: By considering the third order difference equation

$$(i)\Delta^2\left(3^n\left(\Delta x_n\right)^3\right)+\frac{25}{4}3^{n+3}x_{n+1}^3=0$$

has the oscillatory solution $\frac{(-1)^n}{2^n}$, and deviating argument 1. Here $a_n=1, b_n=3^n, c_n=\frac{25}{4}(3^{n+3}),\alpha=1,\&\beta=\mu=3.$

$$(ii)\Delta\left(2n\left(\Delta^2 x_n\right)\right)+8(2n-1)x_{n+3}=0 \tag{5.4}$$

Has deviating argument 3 and $a_n=2n, b_n=1, c_n=8(2n-1),\alpha=\beta=\mu=1.$ Hence $x_n=\frac{1}{3^n}$ is negative solution of (5.4).

Example 5.5: Suppose that $a_n=1, b_n=n+1, c_n=\frac{2(4n^3+21n^2+27n+1)}{(n+1)(n+2)}.$ Let $\alpha=\beta=1$

Then (1.1) becomes,

$$\Delta^2\left(n+1(\Delta x_n)\right)+\frac{2(4n^3+21n^2+27n+1)}{(n+1)(n+2)}x_n=0 \tag{5.5}$$

$$\lim_{n\to\infty}\frac{(-1)^n}{2n}\in R \ \& \ \sum_{n=1}^{\infty}c_n=\infty.$$

Hence by theorem 4.3, $\frac{(-1)^n}{2n}$ is almost oscillatory solution of (5.5).

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