May – June 2020
ISSN: 0193-4120 Page No. 9507 – 9521

On Oscillation of Fractional Functional Differential Equations with Constant Coefficients

S. Panigrahi[1] ,S. Chand[2,*]
1 Department of Mathematics, Siksha 'O' Anusandhan University,
Khandagiri Square, Bhubaneswar−751030, Odisha, India
2 Centre for Applied Mathematics and Computing, Siksha 'O' Anusandhan University,
Khandagiri Square, Bhubaneswar−751030, Odisha, India

Article Info
Volume 83
Page Number: 9507 – 9521
Publication Issue:
May - June 2020

Article History
Article Received: *19 November 2019*
Revised: *27 January 2020*
Accepted: *24 February 2020*
Publication: *18 May 2020*

Abstract:
In this paper the qualitative behaviour of solutions of fractional functional differential and forced fractional functional differential equations with constant coefficients as well as constant positive and negative coefficients have been studied. The Riemann-Liouville definition of fractional order derivatives have been used throughout the paper. Several results have been obtained related to the oscillatory behaviour of solutions of such type of fractional functional differential equations and they have been justified with suitable examples.

Keywords*: Oscillation; Fractional differential equation ; Functional differential equation ; Riemann-Liouville fractional derivative.*
Mathematics Subject Classification (2010): 34A08; 34C10

Introduction:

Fractional differential equations have aroused the interest of researchers since the last few centuries due to their widespread applications in various fields such as electrochemistry, electromagnetic field theories, control theory, fluid flow, optics and signal processing etc. Several researchers such as Chen [[1],[2]], Chen et. al.[3], Liu et. al.[4] have discussed and analyzed the oscillatory behaviour of solutions of differential equations of fractional order.

The qualitative behaviour of functional differential equations have been throughly discussed by Gyori and Ladas[5] and Ladde et. al.[6]. Furthermore, Jaros and Stavroulakis[7], Kon et. al.[8], Li[9], Philos[10], Sficas and Stavroulakis[11], Tang and Yu[12], Dorociakova et. al [13], Ladas and Qian[14], Elabbasy et.al[15] and Elabbasy and Saker [16] have studied the oscillation of delay differential equations with positive and negative coefficients.

Recently mathematicians have started investigating fractional order functional differential equations both qualitatively as well as quantitatively. Lu and Cen[17], Bolat[18], Zhu and Xiang[19] etc. have extensively studied the oscillation criteria for fractional order delay differential equations.

The main aim or objective of our work is to study the qualitative behaviour of solutions of fractional order functional differential equations with constant coefficients. In the present work the oscillation of fractional functional differential equations with constant coefficients as well as fractional functional differential equations with constant positive and negative coefficients have been discussed. Several types of fractional functional differential equations have been investigated for their oscillatory behaviour and the results have been justified by suitable examples.

Our work is motivated by the works of Gyori and Ladas[5], Ladas and Qian[14], Elabbasy et. al[15], Elabbasy and Saker[16], Lu and Cen[17], Bolat[18] and Zhu and Xiang[19]. As it is not always easy to find the actual solution of fractional functional differential equations the qualitative study of the solutions of these type of equations is of vast importance. As the mathematical models of different real life problems give rise to fractional functional differential equations the qualitative study of these type of equations will enable us to understand and analyze the behaviour of the solutions of these type of models.

The paper is organized in the following way. Section 1 is the introduction to the paper. In section 2 some definitions and basic results are given, which are required in the present work. Section 3 contains the main results and the conclusion is given in section 4.

2 Definition and Basic Results

In this section some definitions and basic results are given which will be used in our work.

Definition 2.1:[18] The Riemann Liouville fractional order derivative is defined as follows

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau$$

where $\alpha \in R, n-1 < \alpha < n, n \in N$ and f is a continuous function.

Definition 2.2:[5] Let $x: [0, \infty) \longrightarrow R$ be a real valued function with Laplace transform

$$X(s) = \int_0^{\infty} e^{-st} x(t) dt \qquad (1)$$

If there exists a real number σ_0 such that (1) converges for all s with $Res > \sigma_0$ and diverges for all s with $Res < \sigma_0$ then σ_0 is called the abscissa of convergence of $X(s)$.

Theorem 2.1:[5] Let $x \in C[[0, \infty), R^+]$ such that the abscissae of convergence σ_0 of the Laplace transform $X(s)$ of $x(t)$ is finite. Then $X(s)$ has a singularity at $s = \sigma_0$, i.e., there exists a sequence

$$s_n = \alpha_n + i\beta_n \quad for \quad n = 1,2, \ldots$$

such that

$$\alpha_n \geq \alpha_0 \text{ for } n \geq 1, \lim_{n\to\infty} \alpha_n = \sigma_0, \lim_{n\to\infty} \beta_n = 0 \text{ and } \lim_{n\to\infty} |X(s_n)| = \infty.$$

TEST Engineering & Management

May – June 2020
ISSN: 0193-4120 Page No. 9507 – 9521

Theorem 2.2 :[17] Let

$$y^{\delta}(t) + ay(t-\tau) - by(t-\sigma) = 0 \qquad (2)$$

be a fractional delay differential equation with $a, b, \tau, \sigma \in R^{+}$. Then

$$\lambda + ae^{-\lambda\tau} - be^{-\lambda\sigma} = 0 \qquad (3)$$

is the characteristic equation of (3). Then the following conditions are equivalent.

(i) Every solution of (1) oscillates.

(ii) The characteristic equation of (2) has no real roots.

a. 3 Main Results

Theorem 3.1: Let $a_i, \tau_i \geq 0$ for $i = 1,2,\ldots,n$ and $0 < \delta < 1$, where $\delta = \frac{oddinteger}{oddinteger}$. Then

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \left(\sum_{i=1}^{n} \tau_i\right) > \frac{1}{e}(\lambda^{\delta-1})$$

is a sufficient condition for the oscillation of all solutions of the fractional delay differential equation(FDDE)

$$y^{\delta}(t) + \sum_{i=1}^{n} a_i y(t-\tau_i) = 0. \qquad (4)$$

Proof: The characteristic equation of (4) is given by

$$\lambda^{\delta} + \sum_{i=1}^{n} a_i e^{-\lambda\tau_i} = 0 \qquad (5)$$

By applying the inequality for arithmatic mean and geometric mean, i.e.,

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i$$

and the condition $e^{x} \geq ex$ for $x \geq 0$ we get that for $\lambda < 0$,

$$\lambda^{\delta} + \sum_{i=1}^{n} a_i e^{-\lambda\tau_i} \geq \lambda^{\delta} + n\left(\prod_{i=1}^{n} a_i e^{-\lambda\tau_i}\right)^{\frac{1}{n}}$$

May – June 2020
ISSN: 0193-4120 Page No. 9507 – 9521

$$\geq \lambda^{\delta} + n(\Pi_{i=1}^{n} a_i)^{\frac{1}{n}} e\left(-\frac{\lambda}{n} \sum_{i=1}^{n} \tau_i\right)$$

$$= -\lambda e\left[-\frac{1}{e}\lambda^{\delta-1} + (\Pi_{i=1}^{n} a_i)^{\frac{1}{n}} (\sum_{i=1}^{n} \tau_i)\right] > 0$$

which shows that (5) has no negative roots and furthermore as (5) has no positive roots either, every solution of (4) oscillates.

Example 3.1 : Consider the FDDE

$$y^{\frac{1}{3}}(t) + \frac{3}{2} y(t - \frac{5\pi}{6}) + \frac{1}{2} y(t - \frac{\pi}{6}) = 0 \qquad (6)$$

which satisfies the conditions of Theorem 3.1 as

$$(a_1 a_2)^{\frac{1}{2}} (\tau_1 + \tau_2) = \frac{\sqrt{3}}{2}\pi > \frac{1}{e}(\lambda^{\delta-1}).$$

$y(t) = cost$ is an oscillatory solution of (6).

Theorem 3.2: Let the characteristic equation (5) of the FDDE (4) has a real root. Then there exists an $\alpha_0 \in (0,1)$ such that for every $\alpha \in [0, \alpha_0]$ the FDDE

$$y^{\delta}(t) + \sum_{i=1}^{n} (1-\alpha) a_i y(t - \tau_i) = 0, t \geq 0, \quad 0 < \delta < 1 \qquad (7)$$

has a positive solution.

Proof: The characteristic equation of (7) is given by

$$\nu^{\delta} + \sum_{i=1}^{n} (1-\alpha) a_i e^{-\nu \tau_i} = 0 \qquad (8)$$

which has a real root. It is enough to show that the following statements (i) and (ii) are equivalent.

(i) Eq(5) has no real root.

(ii)There exists an $\alpha_0 \in (0,1)$ such that (8) has no real root for all $\alpha \in [0, \alpha_0]$.

(ii)⇒ (i) is obvious.

To prove (i)⇒(ii). Let

TEST Engineering & Management

May – June 2020
ISSN: 0193-4120 Page No. 9507 – 9521

$$G(\lambda) = \lambda^{\delta} + \sum_{i=1}^{n} a_i e^{-\lambda\tau_i}$$

We see that $G(\infty) = \infty$.

As $G(\lambda) = 0$ has no real root, we have that $G(\lambda) > 0$ for $\lambda \in R$. Furthermore,

$G(-\infty) = \infty$, otherwise (5) has a real root. Therefore the $min_{\lambda \in R} G(\lambda) = \nu$ exists and is

positive. So $G(\lambda) \geq \nu$ for $\lambda \in R$. Let $H(\nu) = \nu^{\delta} + \frac{1}{2}\sum_{i=1}^{n} a_i e^{-\nu\tau_i}, \nu \in R$.

We see that $H(\infty) = H(-\infty) = \infty$. So there exists some $\epsilon > 0$ such that $H(\nu) > 0$ for

$|\nu| > \epsilon$. Choosing $\alpha_0 \in (0, \frac{1}{2})$ such that

$$\alpha_0 \sum_{i=1}^{n} a_i e^{\epsilon\tau_i} \leq \frac{\gamma}{2}$$

we claim that for every $\alpha \in [0, \alpha_0]$, (8) has no real roots. So if $\nu \in R$ with $|\nu| > \epsilon$ then

$$\nu^{\delta} + \sum_{i=1}^{n} (1-\alpha) a_i e^{-\nu\tau_i} \geq \nu^{\delta} + \frac{1}{2}\sum_{i=1}^{n} a_i e^{-\nu\tau_i} > 0.$$

On the other hand for $-\epsilon \leq \nu \leq \epsilon$,

$$\nu^{\delta} + \sum_{i=1}^{n} (1-\alpha) a_i e^{-\nu\tau_i} \geq \frac{\nu}{2} > 0.$$

Hence proved.

Example 3.2: It can easily be verified by choosing $\alpha = 0.5$ and forming the FDDE

$$y^{\frac{1}{3}}(t) + \frac{3}{4} y\left(t - \frac{5\pi}{6}\right) + \frac{1}{4} y\left(t - \frac{\pi}{6}\right) = 0 \tag{9}$$

from the FDDE (6), that every solution of (9) is oscillatory and $y(t) = sin(t - \frac{\pi}{2})$ is an

oscillatory solution of (9).

Theorem 3.3: Let

$$y^{\delta}(t) + ay(t-\tau) = f(t) \tag{10}$$

be a forced FDDE with $0 < \delta < 1$, $a, \tau \in R^{+}$. If

$$\lim_{s \to -\infty} F(s) = 0$$

and $s^{\alpha} + ae^{-s\tau} \neq 0$ then every solution of (10) oscillates.

Proof: Let on the contrary (10) has an eventually positive solution $y(t)$, that is let $y(t) > 0$ for $t \geq -\tau$. So there exists constants M and μ such that

$$|y(t)| \leq Me^{\mu t}, t \geq -\tau.$$

Thus, the Laplace transform

$$Y(s) = \int_0^{\infty} e^{-st} y(t)dt \tag{11}$$

exists for $Res > \mu$.

By taking the Laplace transform of both sides in (10) we get

$$s^{\delta}Y(s) - [D^{\delta-1}y(t)]_{t=0} + ae^{-s\tau}Y(s) + ah(s\tau) = F(s),$$

where $L\{y(t)\} = Y(s)$, $L\{f(t)\} = F(s)$, $e^{-s\tau}\int_{-\tau}^{0} e^{-st} y(t)dt = h(s\tau)$.
So

$$(s^{\delta} + ae^{-s\tau})Y(s) = F(s) - ah(s\tau) + [D^{\delta-1}y(t)]_{t=0}$$

$$\Rightarrow G(s)Y(s) = Q(s) \tag{12}$$

where $G(s) = s^{\alpha} + ae^{-s\tau}$ and $Q(s) = F(s) - ah(s\tau) + [D^{\delta-1}y(t)]_{t=0}$.

Clearly $G(s)$ and $Q(s)$ are entire functions and $G(s) \neq 0$ for all real s. So from (12)

$$Y(s) = \frac{Q(s)}{G(s)}, \quad Res > \sigma_0$$

where σ_0 is the abscissae of convergence of $Y(s)$, i.e., $\sigma_0 = \inf\{\sigma \in R: Y(\sigma) exists\}$.

We claim that $\sigma_0 = -\infty$. Otherwise $\sigma_0 > -\infty$ and by Theorem 2.1 the point $s = \sigma_0$ must be a singularity of $\frac{Q(s)}{G(s)}$. But as this quotient has no singularity on the real axis $\sigma_0 = -\infty$ and

$$Y(s) = \frac{Q(s)}{G(s)} \qquad \forall s \in R \tag{13}$$

Furthermore, we can observe that as $s \to -\infty$, (13) gives us a contradiction as $Y(s)$ and $G(s)$ are always positive where as $Q(s)$ becomes eventually negative. Hence proved.

May – June 2020
ISSN: 0193-4120 Page No. 9507 – 9521

Example 3.3 : Consider the forced FDDE of the form

$$D^{\frac{1}{3}}y(t) + y(t - \tfrac{\pi}{6}) = \sqrt{3}cost \quad (14)$$

which satisfies the conditions of Theorem 3.3 and $y(t) = cost$ is an oscillatory solution of (14).

Theorem 3.4 : Let

$$y^{\delta}(t) + \textstyle\sum_{i=1}^{n} a_i y(t - \tau_i) = f(t) \quad (15)$$

be a forced delay differential equation of fractional order with $0 < \delta < 1$, $a_i, \tau_i \in R^+$, $a_i \neq 0$. If $\lim_{s\to-\infty} F(s) = 0$, where $F(s) = L\{f(t)\}$ as well as $s^{\delta} + \sum_{i=1}^{n} a_i e^{-s\tau_i} \neq 0$, then every solution of (15) oscillates.

Proof: The proof is similar to that of Theorem 3.3.

Example 3.4 : Consider the forced FDDE of the form

$$D^{\frac{1}{3}}y(t) + 2y(t - \tfrac{5\pi}{6}) + y(t - \tfrac{\pi}{6}) = sint \quad (16)$$

which satisfies all the conditions of Theorem 3.4 and $y(t) = cost$ is an oscillatory solution of (16).

Theorem 3.5 : Let

$$y^{\delta}(t) + ay(t - \tau) - by(t - \sigma) = 0 \quad (17)$$

be a FDDE with $0 < \delta < 1$ and $a, b, \tau, \sigma \in R^+$. Then $a > b$ and $\tau \geq \sigma$ is a necessary condition for all solutions of (17) to oscillate.

Proof: The characteristic equation of (17) is given by

$$G(\lambda) = \lambda^{\delta} + ae^{-\lambda\tau} - be^{-\lambda\sigma} = 0 \quad (18)$$

Let every solution of (17) oscillate. So (17) has no real roots. As $G(\infty) = \infty$, it follows that

$$G(0) = a - b > 0 \Rightarrow a > b.$$

May – June 2020
ISSN: 0193-4120 Page No. 9507 – 9521

Also $\tau \geq \sigma$ because if $\tau < \sigma$ and $b > 0$ then $G(-\infty) = -\infty$, which is a contradiction.

Example 3.5 : Consider the following FDDE

$$y^{\frac{1}{3}}(t) + \frac{3}{2}y(t - \frac{5\pi}{6}) - \frac{1}{2}y(t - \frac{5\pi}{6}) = 0 \qquad (19)$$

$a = \frac{3}{2} > \frac{1}{2} = b, \tau = \sigma = \frac{5\pi}{6}$.

It can be easily verified that $y(t) = cost$ is one of its solutions.

Theorem 3.6 : If $a > b, \tau \geq \sigma, b(\tau - \sigma) \leq \lambda^{\delta-1}$ and $(a - b)\tau > \frac{1}{e}[\lambda^{\delta-1} - b(\tau - \sigma)]$

then all solutions of (17) oscillate.

Proof: Let on the contrary all solutions of (17) do not oscillate. So (18) has a real root say λ_0.

Now,

$$\lambda_0(\lambda_0^{\delta-1} - b\int_\sigma^\tau e^{-\lambda_0 s} ds)$$

$$= \lambda_0^\delta + b(e^{-\lambda_0\tau} - e^{-\lambda_0\sigma})$$

$$= -(a - b)e^{-\lambda_0\tau} \leq 0 \qquad (20)$$

For $\lambda \geq 0$

$$\lambda_0^{\delta-1} - b\int_\sigma^\tau e^{-\lambda s} ds \geq \lambda_0^{\delta-1} - b\int_\sigma^\tau ds$$

$$= \lambda_0^{\delta-1} - b(\tau - \sigma) \geq 0 \qquad (21)$$

So $\lambda_0 < 0$.

Thus (20) can be written as

$$\lambda_0[\lambda_0^{\delta-1} - b(\tau - \sigma)] + (a - b)e^{-\lambda_0\tau} < 0$$

$$\Rightarrow \lambda_0 + \frac{a-b}{\lambda_0^{\delta-1} - b(\tau-\sigma)} e^{-\lambda_0\tau} < 0 \qquad (22)$$

So the equation

$$F(\lambda) = \lambda + \frac{a-b}{1-b(\tau-\sigma)} e^{-\lambda\tau} = 0$$

has a real root in $(\lambda_0, 0)$ as $F(\lambda_0) < 0$ by (22) and $F(0) = \frac{a-b}{1-b(\tau-\sigma)} > 0$.

May – June 2020
ISSN: 0193-4120 Page No. 9507 – 9521

Therefore

$$(a-b)\tau \leq \frac{1}{e}[\lambda_0^{\delta-1} - b(\tau-\sigma)]$$

which contradicts our assumption. So all solutions of (17) oscillate.

Example 3.6 : Consider the FDDE given by (19) which satisfies all the assumptions of Theorem 3.6 as well and has the oscillatory solution $x(t) = cost$.

Theorem 3.7 : Let

$$y^{\delta}(t) + ay(t-\tau) - by(t-\sigma) = f(t), 0 < \delta < 1 \tag{23}$$

be a forced FDDE with $a, b, \tau, \sigma \in R^{+}$. If $a > b$, $\tau \geq \sigma$ and $\lim_{s\to-\infty} F(s) = 0$ as well as $s^{\delta} + ae^{-s\tau} - be^{-s\sigma} \neq 0$ then every solution of (23) oscillates.

Proof: Similar to the proof of Theorem 3.3.

Example 3.7 : The following forced FDDE given by

$$D^{\frac{1}{3}}y(t) + 3y(t-\tfrac{5\pi}{6}) - y(t-\tfrac{\pi}{6}) = \frac{-3\sqrt{3}}{2}sint - \frac{1}{2}cost \tag{24}$$

satisfies all the conditions of Theorem 3.9 and $y(t) = sint$ is an oscillatory solution of (24).

Theorem 3.8 : Let

$$y^{\delta}(t) + \sum_{i=1}^{m} a_i y(t-\tau_i) - \sum_{j=1}^{n} b_j y(t-\sigma_j) = f(t), 0 < \delta < 1 \tag{25}$$

be a forced delayed differential equation of fractional order. If $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, $\tau_i \geq \sigma_j$ for all $i = 1, \dots, m$, $j = 1, \dots, n$, $\lim_{s\to-\infty} F(s) = 0$ and $s^{\delta} + \sum_{i=1}^{m} a_i e^{-s\tau_i} - \sum_{j=1}^{n} b_j e^{-s\sigma_j} \neq 0$, then every solution of (25) oscillates.

Proof: The proof is similar to that of Theorem 3.3.

The following example justifies the above theorem.

Example 3.8 : Consider the forced FDDE of the form

$$D^{\frac{1}{3}}y(t) + y(t - \frac{17\pi}{6}) - \frac{1}{5}y(t-\pi) - \frac{1}{2}y(t-2\pi) = \frac{-3}{10}sint \qquad (26)$$

It can be easily verified that (26) satisfies all the conditions of Theorem 3.10 and $y(t) = sint$ is an oscillatory solution of (26).

Theorem 3.9 : Consider the following FDDE

$$D^{\delta}\left[y(t) + D^{\beta}y(t)\right] + ay(t-\tau) = 0 \qquad (27)$$

where $0 < \delta < 1$, $0 < \beta < 1$, $a \in R$, $\tau \in R^{+}$. If $s^{\delta+\beta} + s^{\delta} + ae^{-s\tau} \neq 0$ then every solution of (27) oscillates.

Proof: Let on the contrary (27) has an eventually positive solution $y(t)$. So $y(t) > 0$ for $t \geq -\tau$. So taking the Laplace transform of (27) we get

$$\left(s^{\delta+\beta} + s^{\delta} + ae^{-s\tau}\right)Y(s) = s^{\delta}\left[D^{\beta-1}y(t)\right]_{t=0} + \left[D^{\delta-1}(D^{\beta}y(t) + y(t))\right]_{t=0} - ah(s\tau)$$

where $Y(s) = L\{y(t)\}$ and $h(s\tau) = e^{-s\tau}\int_{-\tau}^{0} e^{-st}y(t)dt.$

Proceeding as in Theorem 3.3 , we get a contradiction. So every solution of (27) oscillates.

Example 3.9: It can be easily verified that the FDDE

$$D^{\frac{1}{2}}\left[y(t) + D^{\frac{1}{3}}\mathrm{y(t)}\right] + 2cos\frac{\pi}{12}y\left(t - \frac{2\pi}{3}\right) = 0 \qquad (28)$$

satisfies all the conditions of Theorem 3.9 and $y(t) = sint$ is an oscillatory solution of (28).

The following theorems can easily be proved.

Theorem 3.10: If

$$D^{\delta}\left[y(t) + D^{\beta}y(t)\right] + ay(t-\tau) = f(t) \qquad (29)$$

where $0 < \delta < 1$, $0 < \beta < 1$, $a \in R$, $\tau \in R^{+}$ is a forced FDDE such that $s^{\delta+\beta} + s^{\delta} + ae^{-s\tau} \neq 0$ and $\lim_{s\to-\infty}F(s) = 0$ holds where $F(s) = L\{f(t)\}$ then every solution of (29) oscillates.

TEST Engineering & Management

May – June 2020
ISSN: 0193-4120 Page No. 9507 – 9521

Theorem 3.11: Let us consider a FDDE with positive and negative coefficients of the form

$$D^{\delta}\left[y(t)+D^{\beta}y(t)\right]+ay(t-\tau)-by(t-\sigma)=0 \qquad (30)$$

where $0<\delta<1$, $0<\beta<1$, $a,b,\tau,\sigma\in R^{+}$. If $s^{\delta+\beta}+s^{\delta}+ae^{-s\tau}-be^{-s\sigma}\neq 0$ then every solution of (30) oscillates.

Theorem 3.12: If

$$D^{\delta}\left[y(t)+D^{\beta}y(t)\right]+ay(t-\tau)-by(t-\sigma)=f(t) \qquad (31)$$

where $\delta,\beta\in(0,1)$, $a,b,\tau,\sigma\in R^{+}$ is a forced FDDE with positive and negative coefficients such that $s^{\delta+\beta}+s^{\delta}+ae^{-s\tau}-be^{-s\sigma}\neq 0$ and $\lim_{s\to-\infty}F(s)=0$ then every solution of (31) oscillates.

The above theorems are justified by the following examples.

Example 3.10: Consider the forced FDDE

$$D^{\frac{1}{2}}\left[y(t)+D^{\frac{1}{3}}y(t)\right]+\sqrt{3}cos\frac{\pi}{12}y(t-\frac{\pi}{2})=cos\frac{\pi}{12}.sint \qquad (32)$$

which satisfies all the conditions of Theorem 3.10 and $y(t)=sint$ is an oscillatory solution of (32).

Example 3.11: Let

$$D^{\frac{1}{2}}\left[y(t)+D^{\frac{1}{3}}y(t)\right]+cos\frac{\pi}{12}y(t-\pi)-\sqrt{3}cos\frac{\pi}{12}y(t-\frac{3\pi}{2})=0 \qquad (33)$$

be a FDDE. It can easily be shown that (33) satisfies all the conditions of Theorem 3.11 and has the oscillatory solution $y(t)=sint$.

Example 3.12: The forced FDDE

$$D^{\frac{1}{3}}\left[y(t)+D^{\frac{1}{5}}y(t)\right]+2cos\frac{\pi}{20}.cos\frac{13\pi}{60}y(t-\pi)-2cos\frac{\pi}{20}.sin\frac{13\pi}{60}y\left(t-\frac{\pi}{2}\right)= -4cos\frac{\pi}{20}.sin\frac{13\pi}{60}sint \qquad (34)$$

justifies Theorem 3.12 and has the oscillatory solution $y(t) = cost$.

Remark 3.1: Theorems 3.9 to Theorem 3.12 will also hold good if $\alpha = \beta$.

Theorem 3.13: Let us consider a fractional functional differential equation(FFDE)

$$D^{\delta}\left[y(t) + D^{\beta} y(t)\right] + ay(t-\tau) - by(t-\sigma) = 0 \tag{35}$$

where $1 < \delta < 2$, $0 < \beta < 1$, $a, b, \tau, \sigma \in R^{+}$, $\delta = \frac{oddinteger}{oddinteger}$, $\beta = \frac{oddinteger}{oddinteger}$.

If $a > b$, $\tau \geq \sigma$, $\lambda \geq 0$, $\lambda^{\delta-1} + \lambda^{\beta+\delta-1} + \frac{\lambda^{\beta-1}(-\delta)}{\Gamma(1-\delta)} \int_0^t s^{-\delta-1} ds \geq b(\tau - \sigma)$, where

$\frac{\lambda^{\beta-1}(-\delta)}{\Gamma(1-\delta)} \int_0^t s^{-\delta-1} ds \leq 0$ and $(a-b)\tau > \frac{1}{e}\left[\lambda^{\delta-1} + \lambda^{\beta+\delta-1} - b(\tau-\sigma)\right]$, then every solution

of (35) oscillates.

Proof: The characteristic equation of (35) is

$$F(\lambda) = \lambda^{\delta} + \lambda^{\delta+\beta} + \frac{\lambda^{\beta} t^{-\delta}}{\Gamma(1-\delta)} + ae^{-\lambda\tau} - be^{-\lambda\sigma} = 0 \tag{36}$$

Let (36) have a real root λ_0. Then

$$\lambda_0^{\delta} + \lambda_0^{\beta+\delta} + \frac{\lambda_0^{\beta} t^{-\delta}}{\Gamma(1-\delta)} + ae^{-\lambda_0\tau} - be^{-\lambda_0\sigma} = 0 \tag{37}$$

Now

$$\lambda_0\left(\lambda_0^{\delta-1} + \lambda_0^{\beta+\delta-1} + \frac{\lambda_0^{\beta-1}(-\delta)}{\Gamma(1-\delta)}\int_0^t s^{-\delta-1} ds - b\int_{\sigma}^{\tau} e^{-\lambda_0 s} ds\right)$$

$$= \lambda_0^{\delta} + \lambda_0^{\beta+\delta} + \frac{\lambda_0^{\beta} t^{-\delta}}{\Gamma(1-\delta)} + b\left(e^{-\lambda_0\tau} - e^{-\lambda_0\sigma}\right)$$

$$= -ae^{-\lambda_0\tau} + be^{-\lambda_0\sigma} + be^{-\lambda_0\tau} - be^{-\lambda_0\sigma}$$

$$= -(a-b)e^{-\lambda_0\tau} \leq 0 \tag{38}$$

For $\lambda \geq 0$

$$\lambda^{\delta-1} + \lambda^{\beta+\delta-1} + \frac{\lambda^{\beta-1}(-\delta)}{\Gamma(1-\delta)} \int_0^t s^{-\delta-1} ds - b \int_\sigma^\tau e^{-\lambda s} ds$$

$$\geq \lambda^{\delta-1} + \lambda^{\delta+\beta-1} + \frac{\lambda^{\beta-1}(-\delta)}{\Gamma(1-\delta)} \int_0^t s^{-\delta-1} ds - b \int_\sigma^\tau ds$$

$$\geq 0 \tag{39}$$

So $\lambda_0 < 0$ which follows from (38).

From (39)

$$\lambda_0\left(\lambda_0^{\delta-1} + \lambda_0^{\beta+\delta-1} - b(\tau - \sigma)\right) \geq 0.$$

Furthermore from (38) and (39)

$$\lambda_0\left(\lambda_0^{\delta-1} + \lambda_0^{\beta+\delta-1} - b(\tau - \sigma)\right) \leq -(a-b)e^{-\lambda_0 \tau} \leq 0$$

$$\Rightarrow \lambda_0\left(\lambda_0^{\delta-1} + \lambda_0^{\beta+\delta-1} - b(\tau - \sigma)\right) + (a-b)e^{-\lambda_0 \tau} \leq 0$$

$$\Rightarrow \lambda_0 + \frac{(a-b)e^{-\lambda_0 \tau}}{(\lambda_0^{\delta-1} + \lambda_0^{\beta+\delta-1} - b(\tau-\sigma))} \leq 0.$$

Thus the equation

$$\lambda + \frac{(a-b)e^{-\lambda\tau}}{\lambda^{\delta-1} + \lambda^{\beta+\delta-1} - b(\tau-\sigma)} = 0$$

has a real root on (λ_0, ∞) which implies that

$$(a-b)\tau \leq \frac{1}{e}\left[\lambda_0^{\delta-1} + \lambda_0^{\delta+\beta-1} - b(\tau - \sigma)\right] \qquad [5, Theorem 2.2.3]$$

which is a contradiction to our assumption. Hence the theorem holds.

Example 3.13: The FFDE

$$D^{\frac{5}{3}}\left[y(t) + D^{\frac{1}{5}} y(t)\right] + 2cos\frac{\pi}{20}cos\frac{53\pi}{60}y(t-2\pi) - 2cos\frac{\pi}{20}sin\frac{53\pi}{60}y\left(t - \frac{3\pi}{2}\right) = 0,$$

$t \in (2\pi, \infty)$(40)

justifies Theorem 3.13 and $y(t) = sint$ is an oscillatory solution of (40).

Furthermore, if we consider the FFDEs of the type

$$D^{\delta}[y(t)+ay(t-\tau)]+by(t-\sigma)=0 \tag{41}$$

where $0<\delta<1$, $a,b,\tau,\sigma\in R^{+}$ and

$$D^{\delta}[y(t)+ay(t-\tau)]+by(t-\sigma)=f(t) \tag{42}$$

where $0<\delta<1$, $a,b,\tau,\sigma\in R^{+}$ then the following theorems hold.

Theorem 3.14: Every bounded solution of (41) oscillates if and only if its characteristic equation has no roots in $(-\infty,0]$.

Theorem 3.15: If $\lim_{s\to-\infty}F(s)=0$, where $F(s)=L\{f(t)\}$ and $s^{\delta}+as^{\delta}e^{-s\tau}+be^{-s\sigma}\neq 0$, $s\in(-\infty,0]$ then every bounded solution of (42) oscillates.

The above two theorems are justified by the following examples.

Example 3.14: The FFDE

$$D^{\frac{1}{2}}[y(t)+2y(t-\pi)]+y(t+\tfrac{\pi}{4})=0 \tag{43}$$

illustrates Theorem 3.14 and has an oscillatory solution $y(t)=sint$.

Example 3.15: The FFDE

$$D^{\frac{1}{2}}[y(t)+y(t-\pi)]+y(t-2\pi)=sint \tag{44}$$

satisfies all the conditions of Theorem 3.15 and $y(t)=sint$ is an oscillatory solution of (44).

Conclusion

In this paper, several results on the oscillation criteria of different types fractional functional differential equations and forced fractional delay differential equations with constant coefficients have been established. We further intend to extend our work to the study of the qualitative behaviour of solutions of systems of fractional functional differential equations in future as well their application to real life problems.

References

1. D. Chen, *Oscillation criteria of fractional differential equations*, Advances in Difference Equations, 33(2012) 1-18.
2. D. Chen, *Oscillatory behavior of a class of fractional differential equations with damping*, UPB Scientific Bulletin, Series A, 75(1)(2013) 107-118.
3. D. Chen, P. Qu and Y. Lan, *Forced oscillation of certain fractional differential equations*, Advances in Difference Equations, 125 (2013) 1-10.

May – June 2020
ISSN: 0193-4120 Page No. 9507 – 9521

4. T. Liu, B. Zheng and F. Meng, *Oscillation on a class of differential equations of fractional order*, Mathematical Problems in Engineering, 2013(2013), Article ID 830836, 13 pages.
5. I. Gyori and G. Ladas, *Oscillation theory of delay differential equation with applications*, Clarendon Press, Oxford, 1991.
6. G. Ladde, V. Lakshmikantham and B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Marcel Dekker, New York, 1987.
7. J. Jaros and I.P. Stavroulakis, *Oscillation tests for delay equations*, Rocky Mountain Journal of Mathematics, 29(1)(1999) 197-207.
8. M. Kon, Y.G. Sficas, and I.P. Stavroulakis, *Oscillation criteria for delay equations*, American Mathematical Society, 128(10)(2000) 2989-2997.
9. B. Li, *Oscillation of first order delay differential equations*, Proceeding of the American Mathematical Society, 124(12)(1996) 3729-3737.
10. Ch. G. Philos, *Oscillation for first order linear delay differential equations with variable coefficients*, Funkcialaj Ekvacioj, 35 (1992) 307-319.
11. Y.G. Sficas and I.P. Stavroulakis, *Oscillation criteria for first-order delay equations*, Bull. London Math. Soc. , 35(2003) 239-246.
12. X.H. Tang and J.S. Yu, *Oscillation of first order delay differential equations*, Journal of Mathematical Analysis and Applications, 248(2000) 247-259.
13. B. Dorociakova, R. Chupac and R. Olach, *Notes on oscillation of linear delay differential equations*, Advances in Difference Equations, 162(2018) 1-7.
14. G. Ladas and C. Qian, *Oscillation in differential equations with positive and negative coefficients*, Canad. Math. Bull., 33(4)(1990) 442-451.
15. E.M. Elabbasy, A.S. Hegazi and S.H. Saker, *Oscillation of solutions to delay differential equations with positive and negative coefficients*, Electronic Journal of Differential Equations, 2000(13)(2000) 1-13.
16. E.M. Elabbasy and S.H. Saker, *Oscillation of delay differential equation with several positive and negative coefficients*, Discussiones Mathematicae Differential Inclusions, Control and Optimization, 23(2003) 39-52.
17. Q. Lu and F. Cen, *The oscillation of a class of the fractional-order delay differential equations* , Discrete Dynamics in Nature and Society, 2014(2014), Article ID 272198, 7 pages.
18. Y. Bolat, *On the oscillation of fractional-order delay differential equations with constant coefficients,* Commun Nonlinear Sci Numer Simulat 19(2014) 3988-3993.