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Harmonic Convex and Concave Fuzzy Mappings with Differentiability

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Abstract:
This research we specifically examined harmonic convex (H-convex) and harmonic concave (H-concave) fuzzy mappings using the principle of differentiability coined by Goetschel and Voxman, and some interrelationships were obtained between differentiable H-convex fuzzy mappings and generalized convex fuzzy mappings like pseudoconvex (pseudoconcave) and quasiconvex (quasiconcave) fuzzy mappings. In addition, differentiable H-convex fuzzy mappings were used to research the KKT conditions for the Harmonic convex fuzzy programming problem (HCFP), the effects of duality, and minmax. Additionally, the findings were explained with sufficient instances.

Keywords: Duality results; Fuzzy optimization; H-convex and H-concave fuzzy mappings; KKT conditions; Minmaxproblem.

1. Introduction

Throughout the qualitative and quantitative dimensions of fuzzy optimization, generalized fuzzy convexity plays an important role. Much research has been done in this area recently and comprehensive analysis has been done on its applicability to mathematical programming problems. Das, Das et al., and Bhargava and Agrawal studied the optimization problems involving H-convex functions. Some researchers such as Iscan and Kunt et al. have recently worked on real-world H-convex functions and used this definition to study the various types of integral inequalities which have many applications in quantum mechanics.Mihai et al. addressed several specific inequalities with respect to harmonic H-convex functions[1], [2].

Modern life scenarios are typically full of confusion and doubt or, in short, the real world is more flippant than smooth. To order to solve the problems arising from this, a new type of problem was considered known as the problem of mathematical programming, which is fuzzy. The main function and any or all of the set of constraints are nonlinear fuzzy functions in some problems or models. There could be bulk transport rates in logistics problems that are cheaper than the usual transport prices. Such levels shall apply if the transported quantity is greater than a certain amount.Similarly, production costs decline in a manufacturing issue as the amount of output increases. In these cases the objective function is nonlinear. Some researchers like Goetschel and Voxman, Nanda and Kar, Syau, Syau and Lee, Wu and Xu, Yan and Xu, Jimenaz et al., Qiu, Gomez et al. and Lizana et al. have studied extensively and explored generalized convex fuzzy mapping and its applications to problem optimization. In the set of fuzzy numbers, Nanda and Kar and Syau studied convexity and quasiconvexity of fuzzy mappings based on the fuzzy-max order.Qiu and Li had defined convex fuzzy mapping with a new metric and gained some related properties[3], [4]. B-preinvexity and B-invexity represent the generalized convexity that Syau introduced. Syau suggested the -convexity and -quasiconvexity of fuzzy mappings based on the limit. Panigrahi et al. considered the restricted fuzzy minimization problem called KKT stationary point fuzzy problem (KTFP) and obtained adequate optimality requirements for KTFP in which the objective function and set of constraints were considered to be differentiable convex fuzzy mappings of several variables.

Cano et al. provided highly generalized differentiability characterizations for convex and invex fuzzy mappings. Gomez et al. studied the necessary and appropriate optimum conditions for problems of optimisation. Jimenez et al. studied vector optimization problems by incorporating the generalized fuzzy convexity principle based on the generalized Hukuhara differentiability definition. Li introduced the idea of b-vex and logarithmic b-vex fuzzy mappings, and examined nonlinear fuzzy optimization issues considering the objective functions and constraints are b-vex fuzzy mappings.Li and Noor obtained multiple preinvex fuzzy mapping characterizations[5]–[8]. Li et al. introduced the idea of univex fuzzy mappings to solve some more problems and obtained a non-dominated solution to the problem of fuzzy optimization, where the objective function and collection of constraints were weakly distinguishable univex fuzzy mappings. Since the definition described by Wu and Xu was rather restrictive, Lizana et al. developed the definition of invex fuzzy mapping based on the highly generalized differentiable fuzzy mapping and used it to research the muddy problems of variational inequality.Lizana et al. studied preinvex fuzzy mappings that were differentiable and twice differentiable. Lizana et al. developed a generalized convexity definition for gH-differentiable fuzzy mappings. Wang and Wu discussed the existence of directional derivative problems for convex fuzzy mappings and the relationship between directional derivative, differential and sub-differential of fuzzy mappings.

While much research has been done on modeling and the solution procedure to a Fuzzy linear programming problem, much less work has been done on the duality of Fuzzy programming issues. Liu et al. focused on the main and dual Fuzzy problems.Bector and Chandra developed and tested a modified pair of Fuzzy primal-dual linear programming problem. The duality theory of fuzzy mathematical programming problems was thoroughly examined by Wu. Wang and Wu studied the characterization of the directional derivative and the differential of two crisp functions that the fuzzy mappings determined. In addition, they applied these results to problems in convex fuzzy programming.[9]–[12]. Nehi and Daryab considered the set of fuzzy mappings to be a partial order relationship and obtained the Fritz John constraint qualification and KKT conditions required for the fuzzy optimization problems with fuzzy coefficients.By using differentiable convex fuzzy mappings with KKT optimal condition, they addressed the optimal conditions of the saddle point in a problem of fuzzy optimisation. Minmax problem is a subset of problem of optimization that has been studied by several writers at. Schmitendorf provided ample conditions of optimality for a minmax problem with or without the principle of convexity. Mishra and Das addressed minimax and symmetric duality using the H-convex function principle for the problem of non-linear integer programming and mixed integer programming, respectively.Ahmed obtained ample optimum conditions and results of duality for the minmax problem of fractional programming, assuming the functions involved were generalized convexly. Minasian introduced a new class of generalized invex functions and, using the new definition of generalized invexity, defined adequate optimal conditions for minimax fractional programming.

Parida et al. studied harmonic fuzzy convexity and obtained several significant findings in connection with this. Parida and Chand introduced the idea of strictly and semi-strictly H-concave fuzzy mappings and obtained numerous related important properties.In particular, since weighted harmonic mean is used to describe these generalized H-convexities, it may be useful in solving mathematical and statistical problems, electrical circuits, economics, genetic algorithms, etc.

In this paper, under the notion of differentiability we have tried to study H-convex (H-concave) fuzzy mappings and used this to study different types of optimization problems. The main aim of these new concepts was to weaken the convexity limitations and directly derive the necessary KKT-type criteria for the non-linear programming problem and use it in results of duality and minmax question[13]–[16].

The document is structured according to the following. In section 2 some known definitions are given regarding fuzzy numbers and differentiable fuzzy mappings. Section 3 considers and addresses the key findings concerning differentiable H- convex (H-concave) fuzzy mappings. In section 4 the necessary conditions of optimality and KKT for Harmonic convex fuzzy programming problem (HUFP) were studied. The findings of duality for HUFP and appropriate conditions for the minmax problem were addressed in section 5 and section 6.

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2. Preliminaries

Zadeh[44] introduced the definition of Fuzzy Convex Sets. For this article, the Euclidean n-dimensional space and the Fuzzy number family are denoted by and respectively. Denotes an open convex set in. Fuzzy numeral space is defined. The following parameterized triple is defined as a fuzzy number,

$$\left\{\left(\tilde{u}_{*}(\alpha), \tilde{u}^{*}(\alpha), \alpha\right) \mid 0 \leq \alpha \leq 1\right\},$$

Where the left and right end points of $[\tilde{u}]_{\alpha}$ are denoted by $\tilde{u}_{*}(\alpha)$ and $\tilde{u}^{*}(\alpha)$.

The α-level set of a fuzzy number $\tilde{u}$ is a closed and bounded interval

$$[\tilde{u}]_{\alpha} = \begin{cases} \{x \in \mathbb{R}^{1} \mid \tilde{u}(x) \geq \alpha\} & if\ 0 < \alpha \leq 1 \\ cl\,(supp\ \tilde{u}), & if\ \alpha = 0 \end{cases}$$

The fuzzy set is a fuzzy number if and only if

(i) $[\tilde{u}]_{\alpha}$ is a closed and bounded interval for each $\alpha \in [0,1]$ and

(ii) $[\tilde{u}]_{1} \neq \phi$.

The addition and nonnegative scalar multiplication of parametrically represented fuzzy numbers are defined. Furthermore if D: $\Gamma \times \Gamma \to [0,\infty)$ is a metric, where Γ is a vector space, then according to Goetschel and Voxman [10], the family of all fuzzy numbers $\tilde{F}_{0}$ can be metricized by D and the ordering „p " on is $\tilde{F}_{0}$ defined for Γ as well as Γ^{*} with $\tilde{F}_{0}^{*} \subset \Gamma^{*}$, where $\tilde{F}_{0}^{*}$ denotes the set of all nonnegative fuzzy numbers. The ordering „p " is a partial ordering on $\tilde{F}_{0}$. Furthermore „p " is a liner ordering for $\tilde{F}_{0}$as any two features in $\tilde{F}_{0}$ are like under the ordering „p ". The differentiability concept of fuzzy mappings are defined.

A fuzzy set $\tilde{f}: M \to \tilde{F}_{0}$ is said to be differentiable at $r_{0} \in \mathbb{R}^{1}$ if there exists $\tilde{f}'(r_{0}) \in \Gamma$ such that

$$\lim_{h \to 0} \frac{\tilde{f}(r_{0}+h) - \tilde{f}(r_{0})}{h} = \tilde{f}'(r_{0})$$

Let $\tilde{f}(r)$ be represented parametrically by $\left\{\left(\tilde{f}_{*}(\alpha,r), \tilde{f}^{*}(\alpha,r), \alpha\right) \mid 0 \leq \alpha \leq 1\right\}$. Then the derivative of the fuzzy mapping $\tilde{f}$ at a point $r_{0} \in \mathbb{R}^{1}$ is given by

$$\nabla_{r}\tilde{f}(r_{0}) = \left\{\left(\tilde{f}_{*r}(\alpha,r_{0}), \tilde{f}^{*}_{r}(\alpha,r_{0}), \alpha\right) \mid 0 \leq \alpha \leq 1\right\},$$

Where $\tilde{f}_{*r}$ and $\tilde{f}^{*}_{r}$ are the partial derivatives of $\tilde{f}_{*}(\alpha,r)$ and $\tilde{f}^{*}(\alpha,r)$ with respect to r.

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Definition 2.1: Let $r_1, r_2 \in M$ and $\lambda \in [0,1]$.

A fuzzy mapping $\tilde{f}: M \to \tilde{F}_0$ is said to be

(i) H-convex (H-concave) if

$$\tilde{f}(\lambda r_1 + (1-\lambda) r_2) \preceq (\succeq) \left[\frac{\lambda}{\tilde{f}(r_1)} + \frac{1-\lambda}{\tilde{f}(r_2)} \right]^{-1}$$

(ii) differentiable convex (concave) at a point r_2 if

$$\tilde{f}(r_1) - \tilde{f}(r_2) \succeq (\preceq) \nabla \tilde{f}(r_2)(r_1 - r_2)$$

(iii) pseudoconvex (pseudoconcave) if for

$$\nabla \tilde{f}(r_2)(r_1 - r_2) \succeq (\preceq) 0 \Rightarrow \tilde{f}(r_1) - \tilde{f}(r_2) \succeq (\preceq) 0$$

(iv) quasiconvex (quasiconcave) if for

$$\tilde{f}(r_1) - \tilde{f}(r_2) \preceq (\succeq) 0 \Rightarrow \nabla \tilde{f}(r_2)(r_1 - r_2) \preceq (\succeq) 0$$

Definition 2.2: [45] Let $M \subseteq \mathbb{R}^n, S \subseteq \mathbb{R}^m$ and $(\bar{r}, \bar{u}) \in M \times S \subseteq \mathbb{R}^n \times \mathbb{R}^m$. $\tilde{f}$ be a fuzzy mapping defined on $M \times S$. the point $(\bar{r}, \bar{u})$ is called a saddle point of $\tilde{f}: M \times S \to \tilde{F}_0$ if

$$\tilde{f}(\bar{r}, u) \preceq \tilde{f}(\bar{r}, \bar{u}) \succeq \tilde{f}(r, \bar{u})$$

for each $(r, u) \in M \times S$.

Theorem 2.3 [45] A point $(\bar{r}, \bar{\lambda}) \in M \times \mathbb{R}^m$ is a saddle point for the fuzzy Lagrangian $\tilde{L}(r, \lambda)$ if and only if the following conditions hold:

(i) $\tilde{L}(\bar{r}, \bar{\lambda}) = \lim_{r \in X} \tilde{L}(r, \bar{\lambda})$

(ii) $\tilde{f}_i(\bar{r}) \preceq 0, i = 1, \ldots, m$

(iii) $\sum_{i=1}^{m} \bar{\lambda} \tilde{f}_i(\bar{r}) = 0$,

where, $\tilde{f}_i: M \to \tilde{F}_0, \lambda_i \geq 0$.

3. Harmonic convexity and generalized convexity of fuzzy mappings

Theorem 3.1 Let $\tilde{f}: M \subset \mathbb{R}^n \to \tilde{F}_0$ be a differentiable fuzzy mapping and $r_1, r_2 \in M$. A necessary and sufficient condition for $\tilde{f}$ to be

(i) H-convex on M is

$$\tilde{f}(r_1) - \tilde{f}(r_2) \succeq (r_1 - r_2)^t \nabla \tilde{f}(r_2) \eta(r_1, r_2) \qquad (3.1)$$

(ii) H-concave on M is

$$\tilde{f}(r_1) - \tilde{f}(r_2) \preceq (r_1 - r_2)^t \nabla \tilde{f}(r_2) \eta(r_1, r_2) \qquad (3.2),$$

where $\eta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\eta(r_1, r_2) = \tilde{f}(r_1) / \tilde{f}(r_2) > 0$.

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$$\lim_{\lambda\to}\frac{\tilde{g}(\lambda r_1+(1-\lambda)r_2)-\tilde{g}(r_2)}{\lambda}=\nabla\tilde{g}(r_2)(r_1-r_2)^t$$

Dividing (3.3) by $\lambda>0$ and taking the limit as $\lambda\to 0$ we have

$$\tilde{g}(r_1)-\tilde{g}(r_2)\preceq(r_1-r_2)^t\nabla\tilde{g}(r_2)$$

$$\Rightarrow\frac{1}{\tilde{f}(r_1)}-\frac{1}{\tilde{f}(r_2)}\preceq=\frac{-\nabla\tilde{f}(r_2)}{\tilde{f}(r_2)^2}(r_1-r_2)^t$$

$$\Rightarrow\tilde{f}(r_1)-\tilde{f}(r_2)\succeq(r_1-r_2)^t\nabla\tilde{f}(r_2)\eta(r_1,r_2),$$

Where $\eta(r_1,r_2)=\tilde{f}(r_1)/\tilde{f}(r_2)$.

Conversely given $\tilde{f}(r_1)-\tilde{f}(r_2)\succeq(r_1-r_2)^t\nabla\tilde{f}(r_2)\eta(r_1,r_2)$

$$\tilde{g}(r_2)-\tilde{g}(r_1)+\nabla\tilde{g}(r_2)(r_1-r_2)^t\succeq 0 \qquad (3.4)$$

Substituting r_2 in place of r_1 and $\lambda r_1+(1-\lambda)r_2$ in place of r_2 in (3.4)

$$\tilde{g}(\lambda r_1+(1-\lambda)r_2)-\tilde{g}(r_2)+\lambda\nabla\left(\tilde{g}(\lambda r_1+(1-\lambda)r_2)\right)(r_1-r_2)^t\succeq 0 \qquad (3.5)$$

Again substituting $\lambda r_1+(1-\lambda)r_2$ in place of r_2 in (3.4)

$$\tilde{g}(\lambda r_1+(1-\lambda)r_2)-\tilde{g}(r_1)+(1-\lambda)\nabla\left(\tilde{g}(\lambda r_1+(1-\lambda)r_2)\right)(r_1-r_2)^t\succeq 0 \qquad (3.6)$$

λ [Equation (3.6)] + $(1-\lambda)$ [Equation (3.5)] gives

$$\tilde{f}(\lambda r_1+(1-\lambda)r_2)\preceq\left[\frac{\lambda}{\tilde{f}(r_1)}+\frac{1-\lambda}{\tilde{f}(r_2)}\right]^{-1}$$

i.e., $\tilde{f}$ is an H-convex fuzzy mapping.

Proof of (ii) is similar to that of (i).

Lemma 3.1 The reciprocal $\tilde{g}$ of a differentiable H-convex (H-concave) fuzzy mapping $\tilde{f}$ is a concave (convex) fuzzy mapping respectively.

Proof: Let $\tilde{f}: M\subset\mathbb{R}^n\to\tilde{F}_0$ be a differentiable H-convex fuzzy mapping and it's reciprocal be $\tilde{g}$. Thus,

$$\tilde{f}(r_1)-\tilde{f}(r_2)\succeq(r_1-r_2)^t\nabla\tilde{f}(r_2)\eta(r_1,r_2),\quad \eta(r_1,r_2)\frac{\tilde{f}(r_1)}{(r_2)}$$

$$\Rightarrow\tilde{g}(r_1)-\tilde{g}(r_2)\preceq(r_1-r_2)^t\nabla\tilde{g}(r_2)$$

So, $\tilde{g}$ is a differentiable concave fuzzy mapping.

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Similarly it can be shown that the reciprocal of a differentiable H-concave fuzzy mapping is a convex fuzzy mapping.

Example 3.1 Let $\tilde{f}:(-1,1)\to \tilde{F}_0$ be a fuzzy mapping defined parametrically as

$$\tilde{f}(r)[\alpha]=\left\{\left(\frac{1+\alpha}{1-r^2},\frac{4-2\alpha}{1-r^2},\alpha\right)\,\middle|\,0\le\alpha\le1\right\}$$

Let $\quad \eta(r,s)=\dfrac{1-s^2}{1-r^2}$

Here $\quad \tilde{f}_*(r,\alpha)=\dfrac{1+\alpha}{1-r^2}$ and $\tilde{f}^*(r,\alpha)=\dfrac{4-2\alpha}{1-r^2}$

Now $\quad \tilde{f}_*(r,\alpha)-\tilde{f}_*(s,\alpha)-(r-s)^t\,\nabla\tilde{f}_*(s,\alpha)\,\eta(r,s)\ge0$

and $\quad \tilde{f}^*(r,\alpha)-\tilde{f}^*(s,\alpha)-(r-s)^t\,\nabla\tilde{f}^*(s,\alpha)\,\eta(r,s)\ge0$

So $\quad \tilde{f}(r)-\tilde{f}(s)\succeq(r-s)^t\,\nabla\tilde{f}(s)\,\eta(r,s)$

i.e., $\tilde{f}$ is a differentiable H-convex fuzzy mapping. Now, the parametric representation of the reciprocal of $\tilde{f}$ is given by

$$\tilde{g}(r)[\alpha]=\left\{\left((1+\alpha)(1-r^2),(4-2\alpha)(1-r^2),\alpha\right)\,\middle|\,0\le\alpha\le1\right\}$$

Here it can be easily shown that $\tilde{g}$ is a differentiable concave fuzzy mapping.

The following theorems can be proved easily by using the definition of differentiable H-convex (H-concave) fuzzy mappings.

Theorem 3.2 (i) Let $\tilde{f}:M\to\tilde{F}_0$ be a differentiable H-convex fuzzy mapping then for $r_1,r_2\in M$, $\tilde{f}$ achieves its global minimum at r_2 on M if $(r_1-r_2)^t\,\nabla\tilde{f}(r_2)\preceq 0$.

(ii) Suppose that $\tilde{f}:M\to\tilde{F}_0$ be a differentiable H-concave fuzzy mapping then for $r_1,r_2\in M$, $\tilde{f}$ achieves its global maximum at r_2 on M if $(r_1-r_2)^t\,\nabla\tilde{f}(r_2)\preceq 0$.

Proof: The proof proceeds as follows.

(i) As $\tilde{f}:M\to\tilde{F}_0$ is a differentiable H-convex fuzzy mapping and $(r_1-r_2)^t\,\nabla\tilde{f}(r_2)\preceq 0$, we get $\tilde{f}(r_1)\succeq\tilde{f}(r_2)$.

i.e., $\tilde{f}$ achieves its global minimum at r_2 on M.

By using the same procedure as above we can show (ii).

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Theorem 3.3 Suppose that $\tilde{f}: M \to \tilde{F}_0$ be a differentiable fuzzy mapping.

(i) If $\tilde{f}$ is H-convex on M then it is a pseudoconvex fuzzy mapping on M.

(ii) If $\tilde{f}$ is H-concave on M then it is a pseudoconcave fuzzy mapping on M.

Proof: According to pseudoconvexity assumption of $\tilde{f}$ we get $(r_1 - r_2)^t \nabla \tilde{f}(r_2) \succeq 0$.

As $\tilde{f}$ is a differentiable H-convex fuzzy mapping on M, so from (3.1) we can write $\tilde{f}(r_1) - \tilde{f}(r_2) \succeq 0$

By using the same procedure as above we can show (ii).

Example 3.2 Let us define a fuzzy mapping $\tilde{f}: M \subset \mathbb{R}^+ \to \tilde{F}_0$ represented parametrically by

$$\tilde{f}(r)[\alpha] = \left\{ \left(\frac{\alpha - 2}{1 + r^2}, \frac{-\alpha}{1 + r^2}, \alpha \right) \mid 0 \le \alpha \le 1 \right\}$$

Here $\tilde{f}_*(r, \alpha) = \dfrac{\alpha - 2}{1 + r^2}, \tilde{f}^*(r, \alpha) = \dfrac{-\alpha}{1 + r^2}$ and $\eta(r, s) = \dfrac{1 + s^2}{1 + r^2}$

It can be easily verified that $\tilde{f}$ is a differentiable H-convex fuzzy mapping. According to the pseudoconvexity assumption.

$$(r - s)^t \nabla \tilde{f}_*(s, \alpha) \ge 0 \text{ and } \eta(r, s) > 0 \Rightarrow \tilde{f}_*(r, \alpha) \ge \tilde{f}_*(s, \alpha)$$

and
$$(r - s)^t \nabla \tilde{f}^*(s, \alpha) \ge 0 \text{ and } \eta(r, s) > 0 \Rightarrow \tilde{f}^*(r, \alpha) \ge \tilde{f}^*(s, \alpha)$$

Theorem 3.4 Suppose that $\tilde{f}: M \to \tilde{F}_0$ be a differentiable fuzzy mapping.

(i) If $\tilde{f}$ is H-convex on M then it is a quasiconvex fuzzy mapping on M.

(ii) If $\tilde{f}$ H-concave on M then it is a quasiconcave fuzzy mapping on M.

Proof: According to quasiconvexity assumption of $\tilde{f}$ we get

$$\tilde{f}(r_1) - \tilde{f}(r_2) \preceq 0$$

As $\tilde{f}$ is a differentiable H-convex fuzzy mapping on M, so from (3.1) we can write

$$(r_1 - r_2)^t - \nabla \tilde{f}(r_2) \preceq 0$$

By using the same procedure as above we can show (ii).

Remark 3.1

The following statements are observed from Theorem 3.1.

(i) If $\eta(r_1, r_2) = 1$, then Theorem 3.1 (i) and Theorem 3.1 (ii) reduce to the usual definition of convexity and concavity respectively but the converse is not true in general.

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If $\rho : M \times M \to ¡$ and $\rho(r_1, r_2) = \eta(r_1, r_2)(r_1 - r_2)$, then Theorem 3.1 (i) reduce to the definition of invexity.

(i) If $b(r_1, r_2) = \dfrac{1}{\eta(r_1, r_2)}$, then Theorem 3.1 (i) reduce to the definition of B-vexity.

The converse part of (i) of Remark 3.1 is justified by the following example.

Example 3.3 Consider a fuzzy mapping $\tilde{f}: M \subseteq ¡ \to \tilde{F}_0$ whose α-level set is defined by

$$\tilde{f}(r)[\alpha] = \left[\left((1+\alpha)(1+r^2), (4-2\alpha)(1+r^2), \alpha\right) \mid 0 \le \alpha \le 1\right]$$

As $$\tilde{f}_*(r) - \tilde{f}_*(s) - \nabla \tilde{f}_*(s)(r-s) \ge 0$$

and $$\tilde{f}^*(r) - \tilde{f}^*(s) - \nabla \tilde{f}^*(s)(r-s) \ge 0$$

So $\tilde{f}$ is a convex fuzzy mapping for all $r, s \in M$.

But for $r = 1$ and $s = 2$ we get

$$\tilde{f}_*(r) - \tilde{f}_*(s) - \nabla \tilde{f}_*(s)(r-s)\eta(r,s) < 0$$

and $$\tilde{f}^*(r) - \tilde{f}^*(s) - \nabla \tilde{f}^*(s)(r-s)\eta(r,s) < 0$$

So, by using Theorem 3.1 we get $\tilde{f}$ is not an H-convex fuzzy mapping.

4. Sufficient optimality conditions and KKT conditions for Harmonic convex fuzzy programming problem (HCFP)

By using the results on H-convex and H-concave fuzzy mappings discussed in Section 3, we have defined a Harmonic convex fuzzy programming problem (HCFP), where the objective function and set of constraints are differentiable H-convex fuzzy mappings. In this section, we have studied sufficient optimality conditions and KKT conditions of non-linear programming problem using differentiable H-convex functions.

Suppose $\tilde{f}_i: M \subset ¡^n \to \tilde{F}_0$; $i = 0, 1, 2\ldots.., m$ are differentiable H-convex fuzzy mappings defined on an open convex set $M \subseteq ¡^n$. The following primal problem is considered as the Harmonic convex fuzzy programming problem (HCFP) and dual problem as the Dual harmonic convex fuzzy programming problem (DHCFP).

HCFP $\quad \min \tilde{f}_0(r)$

subject to $\quad \tilde{f}_i(r) \preceq 0, r \in M \qquad (4.1)$

DHCFP $\quad \max \tilde{\psi}(r,u)$

subject to $$\nabla \tilde{f}_0(r)+\sum_{i=1}^{m} u_i \nabla \tilde{f}_i(r)=0 \qquad (4.2)$$

$$\sum_{i=1}^{m} u_i \tilde{f}_i(r)=0,\ u_i \geq 0, i=1,2,...,m \qquad (4.3)$$

$r \in M$ and $u=(u_1,u_2,......,u_m)^t \in \mathbb{R}^m$,

where, $\tilde{\Psi}(r,u)=\tilde{f}_0(r)+\sum_{i=1}^{m} u_i \tilde{f}_i(r)$ is the Largrangian fuzzy function.

Theorem 4.1 (Sufficient optimality condition for HCFP) Suppose that $\tilde{f}_i$ are H-convex fuzzy mappings and differentiable at $\hat{r} \in M \subset \mathbb{R}^m$ for $i=0,1,2,3,....,m$. If $(\hat{r},\hat{u})$ satisfies the optimality conditions from equations (4.1) to (4.3) then $\hat{r}$ is a solution of HCFP.

Proof: As $\tilde{f}_0$ and $\tilde{f}_i, i=1,2,....,m$ are H-convex and differentiable at $\hat{r}$ we have

$$\tilde{f}_0(r)-\tilde{f}_0(\hat{r}) \succeq (r-\hat{r})^t \nabla \tilde{f}_0(\hat{r})\eta_0(r,\hat{r}) \qquad (4.4)$$

$$\tilde{f}_i(r)-\tilde{f}_i(\hat{r}) \succeq (r-\hat{r})^t \nabla \tilde{f}_i(\hat{r})\eta_i(r,\hat{r}) \qquad (4.5),$$

where $\eta_0(r,\hat{r})>0$ and $\eta_i(r,\hat{r})>0$.

Now

$$\begin{aligned}
\tilde{f}_{0*}(r,\alpha)-\tilde{f}_{0*}(\hat{r},\alpha) &\geq (r-\hat{r})^t \nabla \tilde{f}_{0*}(\hat{r},\alpha)\eta_0(r,\hat{r}) \\
&= -(r-\hat{r})^t \sum_{i=1}^{m} \hat{u}_i \nabla \tilde{f}_{i*}(\hat{r},\alpha)\eta_0(r,\hat{r}) && \text{(by (4.2))} \\
&\geq \sum_{i=1}^{m} \hat{u}_i \left(\tilde{f}_{i*}(\hat{r},\alpha)-\tilde{f}_{i*}(r,\alpha)\right)\frac{\eta_0(r,\hat{r})}{\eta_i(r,\hat{r})} && \text{(by (4.5))} \\
&\geq -\sum_{i=1}^{m} \hat{u}_i \tilde{f}_{i*}(\hat{r},\alpha) && \text{(by (4.3))} \\
&\geq 0 \quad \text{(BY (4.1))}
\end{aligned}$$

Similarly, $\tilde{f}_0^*(r,\alpha)-\tilde{f}_0^*(\hat{r},\alpha) \geq 0$

So, $\tilde{f}_0(r) \succeq \tilde{f}_0(\hat{r})$

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The following example justifies Theorem 4.1.

Example 4.1 Let us consider a fuzzy mapping

$$\text{Min } \tilde{f}(r)$$

subject to $\tilde{g}(r) \preceq 0, r \in (0,1)$ with α - level set

$$\tilde{f}(r)[\alpha] = \left[\left(\frac{\alpha-2}{r_1^2+r_2^2}, \frac{-\alpha}{r_1^2+r_2^2}, \alpha\right) \middle| \, 0 \le \alpha \le 1\right]$$

$$\text{and} \quad \tilde{g}(r)[\alpha] \left[\left((\alpha-2)\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}-1\right), (-\alpha)\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}-1\right), \alpha\right)\right]$$

$$\text{Let} \quad \eta_0(r,s) = \frac{s_1^2+s_2^2}{r_1^1+r_2^2}, \eta_1(r,s) = \frac{(s_1-1)+(s_2-1)^2}{(r_1-1)^2+(r_2-1)^2}$$

$$\zeta(r,s) = (r_1-s, r_2-s_2), r = (r_1,r_2)^t \text{ and } s = (s_1,s_2)^t$$

Thus we observe that $\tilde{f}(r)$ and $\tilde{g}(r)$ are differentiable H-convex fuzzy mappings with respect to ζ , η_0 and ζ, η_1 respectively.

Then for each $\alpha \in [0,1]$

$$\tilde{f}_*(r_1,r_2,\alpha) = \frac{\alpha-2}{r_1^2+r_2^2}, \tilde{f}^*(r_1,r_2,\alpha) = \frac{-\alpha}{r_1^2+r_2^2}$$

$$\tilde{g}_*(r_1,r_2,\alpha) = (\alpha-2)\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}\right)$$

$$\tilde{g}^*(r_1,r_2,\alpha) = (-\alpha)\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}\right)$$

Lagrangian fuzzy function is

$$\tilde{\Psi}(r_1,r_2,u) = \tilde{f}(r_1,r_2) + u\tilde{g}(r_1,r_2)|$$

By using the sufficient optimality criteria we get the following equations

$$\tilde{\Psi}_*(r_1,r_2,u,\alpha) = \tilde{f}_*(r_1,r_2,\alpha) + u\tilde{g}_*(r_1,r_2,\alpha) = \frac{\alpha-2}{r_1^2+r_2^2} + (\alpha-2)u\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}-1\right)$$

$$\tilde{\Psi}^*(r_1,r_2,u,\alpha) = \tilde{f}^*(r_1,r_2,\alpha) + u\tilde{g}^*(r_1,r_2,\alpha) = \frac{-\alpha}{r_1^2+r_2^2} + (-\alpha u)\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}-1\right)$$

$$\nabla_{r_1}\tilde{\Psi}_*(r_1,r_2,u,\alpha)=\frac{(4-2\alpha)r_1}{\left(r_1^2+r_2^2\right)^2}+\frac{(4-2\alpha)(r_1-1)u}{(r_1-1)^2+(r_2-1)^2}$$

$$\nabla_{r_2}\tilde{\Psi}_*(r_1,r_2,u,\alpha)=\frac{(4-2\alpha)r_2}{\left(r_1^2+r_2^2\right)^2}+\frac{(4-2\alpha)(r_2-1)u}{(r_1-1)^2+(r_2-1)^2}$$

$$\nabla_{r_1}\tilde{\Psi}^*(r_1,r_2,u,\alpha)=\frac{2\alpha r_1}{\left(r_1^2+r_2^2\right)^2}+\frac{2\alpha(r_1-1)u}{(r_1-1)^2+(r_2-1)^2}$$

$$\nabla_{r_2}\tilde{\Psi}^*(r_1,r_2,u,\alpha)=\frac{2\alpha r_2}{\left(r_1^2+r_2^2\right)^2}+\frac{2\alpha(r_2-1)u}{(r_1-1)^2+(r_2-1)^2}$$

Now we have to solve the following equations

$$\nabla_{r_1}\tilde{\Psi}_*(r_1,r_2,u,\alpha)=0=\nabla_{r_2}\tilde{\Psi}_*(r_1,r_2,u,\alpha)$$

$$\nabla_{r_1}\tilde{\Psi}^*(r_1,r_2,u,\alpha)=0=\nabla_{r_2}\tilde{\Psi}^*(r_1,r_2,u,\alpha)$$

$$u\tilde{g}_*(r_1,r_2,\alpha)=0=u\tilde{g}^*(r_1,r_2,\alpha)$$

$$\tilde{g}_*(r_1,r_2,\alpha)\le 0,\quad \tilde{g}^*(r_1,r_2,\alpha)\le 0,\ u\ge 0$$

i.e. to solve

$$\frac{(4-2\alpha)r_1}{\left(r_1^2+r_2^2\right)^2}+\frac{(4-2\alpha)(r_1-1)}{(r_1-1)^2+(r_2-1)^2}=0=\frac{(4-2\alpha)r_2}{\left(r_1^2+r_2^2\right)^2}+\frac{(4-2\alpha)(r_2-1)}{(r_1-1)^2+(r_2-1)^2}$$

$$\frac{2\alpha r_1}{\left(r_1^2+r_2^2\right)^2}+\frac{2\alpha(r_1-1)}{(r_1-1)^2+(r_2-1)^2}=0=\frac{2\alpha r_2}{\left(r_1^2+r_2^2\right)^2}+\frac{2\alpha(r_2-1)}{(r_1-1)^2+(r_2-1)^2}$$

$$(\alpha-2)u\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}-1\right)=0=(-\alpha)u\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}-1\right)$$

$$(\alpha-2)\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}-1\right)\le 0,(-\alpha)\left(\frac{1}{(r_1-1)^2+(r_2-1)^2}-1\right)\le 0,$$

$u\ge 0,\ 0\le\alpha\le 1.$

Solvingtheaboveequationswegetthat $r^*=\left(1-\frac{1}{\sqrt{2}},1-\frac{1}{\sqrt{2}}\right)^t$ is the minimum point with

$$u=\frac{1}{\left(\sqrt{2}-1\right)^3}.$$

KKT Conditions for HCFP

Theorem 4.2 Suppose that $X=\{r\in M:\tilde{f}_i(r)\preceq 0, i=1,2,\ldots,m\}$ and $\bar{r}\in X$. If $\tilde{f}_i(r):M\to \tilde{F}_0,\ i=0,1,2,\ldots,m$ are differentiable H-convex fuzzy mappings at $\bar{r}\in M$ and $(\bar{r},\bar{u})\in M\times \mathbb{R}_+^m$ satisfies the optimality conditions (4.2) and (4.3), then $(\bar{r},\bar{u})$ is a saddle point for the fuzzy Lagrangian $\tilde{\Psi}(r,u)$ with $\eta_i(r,\bar{r})=1$ for $i=0,1,\ldots m$. Conversely, If $(\bar{r},\bar{u})\in M\times \mathbb{R}_+^m$ is a saddle point for the fuzzy Lagrangian $\tilde{\Psi}(r,u)$, then $\bar{r}$ is a solution to the HCFP which is optimal in nature and hence satisfies the optimality conditions given in (4.2) and (4.3).

Proof: As $\tilde{f}_0(r)$ and $\tilde{f}_i(r), i=1,2,\ldots,m$ are differentiable H-convex fuzzy mappings at $\bar{r}$ and $\eta_i(r,\bar{r})=1$ for $i=0,1,\ldots,m$ we can write

$$\tilde{f}_0(r)-\tilde{f}_0(\bar{r})\succeq \nabla\tilde{f}_0(\bar{r})(r-\bar{r})^t \qquad (4.6)$$

$$\tilde{f}_i(r)-\tilde{f}_i(\bar{r})\succeq \nabla\tilde{f}_i(\bar{r})(r-\bar{r})^t, i=1,2,\ldots,m$$

$$\Rightarrow \sum_{i=1}^{m}\bar{u}_i\tilde{f}_i(r)-\sum_{i=1}^{m}\bar{u}_i\tilde{f}_i(\bar{r})\succeq \sum_{i=1}^{m}\bar{u}_i\nabla\tilde{f}_i(\bar{r})(r-\bar{r})^t$$

(4.7)

From (4.6) and (4.7) we get

$$\tilde{f}_0(r)+\sum_{i=1}^{m}\bar{u}_i\tilde{f}_i(r)\succeq \tilde{f}_0(\bar{r})+\sum_{i=1}^{m}\bar{u}_i\tilde{f}_i(\bar{r})+\left[\nabla\tilde{f}_0(\bar{r})+\sum_{i=1}^{m}\bar{u}_i\nabla\tilde{f}_i(\bar{r})\right](r-\bar{r})^t$$

$$\Rightarrow \tilde{f}_0(r)+\sum_{i=1}^{m}\bar{u}_i\tilde{f}_i(r)\succeq \tilde{f}_0(\bar{r})+\sum_{i=1}^{m}\bar{u}_i\tilde{f}_i(\bar{r}) \qquad \{\text{by } (4.6)\}$$

$$\Rightarrow \tilde{\Psi}(r,\bar{u})\succeq \tilde{\Psi}(\bar{r},\bar{u})$$

$$\Rightarrow \tilde{\Psi}(\bar{r},\bar{u})=\min_{r\in M}\tilde{\Psi}(r,\bar{u})$$

As $\bar{r}\in X$ and $(\bar{r},\bar{u})$ satisfies (4.3) we have for i=1,2, …., m

$$\tilde{f}_i(\bar{r})\preceq 0 \text{ and } \sum_{i=1}^{m}\bar{u}_i\tilde{f}_i(\bar{r})=0$$

From Theorem 2.3, we get $(\bar{r},\bar{u})$ is a saddle point for the fuzzy Lagrangian $\tilde{\Psi}(r,u)$. Conversely, suppose $(\bar{r},\bar{u})\in M\times\mathbb{R}_+^m$ is a saddle point for the fuzzy Lagrangian $\tilde{\Psi}(r,u)$.

Thus by Definition 2.2 we get

$$\tilde{\Psi}(\bar{r},\bar{u})\preceq \tilde{\Psi}(r,\bar{u})$$

$$\Rightarrow \tilde{f}_0(\bar{r})+\sum_{i=1}^{m}\bar{u}_i\tilde{f}_i(\bar{r})\preceq \tilde{f}_0(r)+\sum_{i=1}^{m}\bar{u}_i\tilde{f}_i(r)$$

$$\Rightarrow \tilde{f}_0(\bar{r}) \preceq \tilde{f}_0(r) + \sum_{i=1}^{m} \bar{u}_i \tilde{f}_i(r) \text{ (by (4.3))}$$

i.e., $\tilde{f}_0(\bar{r}) \preceq \tilde{f}_0(r)$. (As $r \in X$)

Therefore, $\bar{r}$ is an optimal solution to the problem HCFP.

As $\tilde{\Psi}(r,\bar{u})$ is a differentiable H-convex fuzzy mapping and $\bar{r}$ is a minimum of $\tilde{\Psi}(r,\bar{u})$, so by Theorem 3.2 (i) we have

$$\nabla \tilde{\Psi}(r,\bar{u}) = 0, \bar{r} \in M$$

$$\Rightarrow \nabla \tilde{f}_0(\bar{r}) + \sum_{i=1}^{m} \bar{u}_i \nabla \tilde{f}_i(\bar{r}) = 0.$$

Example 4.2 Let us consider the fuzzy mapping defined in Example 4.1.

AsthehypothsesofTheorem4.2aresatisfiedforthefuzzymappings $\tilde{f}(r)$ and $\tilde{g}(r)$,soit canbeseen that $r^* = \left(1-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)^t$ isthesaddlepointoftheminimizationproblemdefined in Example 4.1.

4. Duality results forHCFP

Theorem 5.1 (Weak duality)

Let $\tilde{f}_i(r), i = 0,1,\dots,m$ be differentiable H-convex fuzzy mappings on $M \subseteq \mathbb{R}^n$. If $\tilde{r}$ satisfies the constraints of HCFP and $(\hat{r},\hat{s})$ satisfies the constraints of DHCFP, then $\tilde{f}_0(\tilde{r}) \succeq \tilde{f}_0(\hat{r}), \hat{r} \in M \subseteq \mathbb{R}^n$ and $\hat{s} = (\hat{s}_1, \hat{s}_2, \dots, \hat{s}_m) \in S \subseteq \mathbb{R}^m$.

Proof: As $\tilde{f}_i(r), i = 0,1,\dots,m$ are differentiable and H-convex fuzzy mappings on M, we have for $i = 0$

$$\tilde{f}_0(\tilde{r}) - \tilde{f}_0(\hat{r}) \succeq (\tilde{r} - \hat{r})^t \nabla \tilde{f}_0(\hat{r}) \eta_0(\tilde{r}, \hat{r}), \eta_0(\tilde{r}, \hat{r}) > 0 \quad (5.1)$$

and $\tilde{f}_i(\tilde{r}) - \tilde{f}_i(\hat{r}) \succeq (\tilde{r} - \hat{r})^t \nabla \tilde{f}_i(\hat{r}) \eta_i(\tilde{r}, \hat{r}), \eta_i(\tilde{r}, \hat{r}) > 0$ (5.2)

Furthermore, as $\tilde{r}$ satisfies HCFP

$$\tilde{f}_i(\tilde{r}) \preceq 0, \quad i = 1,\dots,m \quad (5.3)$$

and as $(\hat{r},\hat{s})$ satisfies DHCFP, so for $\hat{r} \in M, \hat{s}_i \in S, \hat{s}_i \geq 0, \quad i = 1,2,\dots,m$

$$\nabla \tilde{f}_0(\hat{r}) + \sum_{i=1}^{m} \hat{s}_i \nabla \tilde{f}_i(\hat{r}) = 0 \quad (5.4)$$

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$$\sum_{i=1}^{m} \hat{s}_i \tilde{f}_i(\hat{r}) = 0 \tag{5.5}$$

From (5.1) and (5.4), we have

$$\tilde{f}_0(\tilde{r}) - \tilde{f}_0(\hat{r}) \succeq -\sum_{i=1}^{m} \hat{s}_i (\tilde{r} - \hat{r})^t \nabla \tilde{f}_i(\hat{r}) \eta_0(\tilde{r}, \hat{r})$$

$$\succeq \sum_{i=1}^{m} \hat{s}_i \left(\tilde{f}_i(\hat{r}) - \tilde{f}_i(\tilde{r})\right) \frac{\eta_0(\tilde{r}, \hat{r})}{\eta_i(\tilde{r}, \hat{r})} \qquad \text{(by (5.2))}$$

$$\succeq \sum_{i=1}^{m} \hat{s}_i \tilde{f}_i(\hat{r}) \qquad \text{(by (5.3))}$$

$$= 0 \quad \text{(by (5.5))}$$

Therefore, $\tilde{f}_0(\tilde{r}) \succeq \tilde{f}_0(\hat{r})$

Example 5.1 We consider the following fuzzy minimization problem which is the dual of the primal given in Example 4.1.

$$max\, \tilde{f}(s)[\alpha] = \left[\left((\alpha-2)\left(\frac{1}{s_1^2+s_2^2}\right), (-\alpha)\left(\frac{1}{s_1^2+s_2^2}\right), \alpha\right) \middle| \, 0 \le \alpha \le 1\right]$$

subject to

$$(4-2\alpha)\left[\frac{s_1}{\left(s_1^2+s_2^2\right)^2} + \frac{(s_1-2)u}{\left((s_1-2)^2+(s_2-2)^2\right)^2}\right] = 0$$

$$(4-2\alpha)\left[\frac{s_2}{\left(s_1^2+s_2^2\right)^2} + \frac{(s_2-2)u}{\left((s_1-2)^2+(s_2-2)^2\right)^2}\right] = 0$$

$$2\alpha\left[\frac{s_1}{\left(s_1^2+s_2^2\right)^2} + \frac{(s_1-2)u}{\left((s_1-2)^2+(s_2-2)^2\right)^2}\right] = 0$$

$$2\alpha\left[\frac{s_2}{\left(s_1^2+s_2^2\right)^2} + \frac{(s_2-2)u}{\left((s_1-2)^2+(s_2-2)^2\right)^2}\right] = 0$$

$$(\alpha-2)u\left[\frac{1}{\left((s_1-2)^2+(s_2-2)^2\right)^2} - 1\right] = 0$$

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$$(-\alpha)u\left[\frac{1}{\left((s_1-2)^2+(s_2-2)^2\right)^2}-1\right]=0,$$

$u\geq 0, s_1, s_2 \in (0,1).$

The point $s=\left(2-\frac{1}{\sqrt{2}},2-\frac{1}{\sqrt{2}}\right), u=\frac{1}{(2\sqrt{2}-1)}$ is a dual feasible solution. As $\tilde{f}$ and $\tilde{g}$ satisfies the hypotheses of Theorem 5.1, the weak duality Theorem holds for any $r\in(0,1)\subset \mathbb{R}^n$.

Theorem 5.2 (Forward Duality)

Let $Y=\left\{(r,s)\mid r\in M\subseteq \mathbb{R}^n, s\in M\subseteq \mathbb{R}^m, \nabla\tilde{f}_0(r)+\sum_{i=1}^{m} s_i\nabla\tilde{f}_i(r)=0, s_i\geq 0\right\}$

Let $\tilde{f}_i(r), i=0,1,2,....,m$ are differentiable H-convex fuzzy mappings at $r\in M$ and $(\bar{r},\bar{s})$ satisfies the optimality criteria (4.3). In particular for $\eta_0(\bar{r},r)=\eta_i(\bar{r},r),\ i=1,2,...,m,$ is $\bar{r}$ solves HCFP, then there exists $\bar{s}=(\bar{s}_1,\bar{s}_2,....,\bar{s}_m)\in R^m$ and $\bar{s}_i\geq 0$ such that $(\bar{r},\bar{s})\in Y$ satisfies DHCFP and $\tilde{\Psi}(\bar{r},\bar{s})=\tilde{f}_0(\bar{r})$.

Proof: Let $\bar{r}$ solve the HCFP. So, $\tilde{f}_i(\bar{r})\preceq 0$.

Furthermore, let (r,s) be an arbitrary element of the set Y. Then

$$\tilde{\Psi}(\bar{r},\bar{s})-\tilde{\Psi}(r,s)=\tilde{f}_0(\bar{r})+\sum_{i=1}^{m}\bar{s}_i\tilde{f}_i(\bar{r})-\tilde{f}_0(r)-\sum_{i=1}^{m}s_i\tilde{f}_i(r)$$

$$=\tilde{f}_0(\bar{r})-\tilde{f}_0(r)-\sum_{i=1}^{m}s_i\tilde{f}_i(r). \qquad \text{[as } (\bar{r},\bar{s}) \text{ satisfies (4.3)]} \qquad (5.6)$$

As $\tilde{f}_0$ is a differentiable H-convex fuzzy mapping at r.

$$\tilde{f}_0(\bar{r})-\tilde{f}_0(r)\succeq(\bar{r}-r)^t\nabla\tilde{f}_0(r)\eta_n(\bar{r},r) \qquad (5.7)$$

From (5.6) and (5.7) we have

$$\tilde{\Psi}_*(\bar{r},\bar{s},\alpha)-\tilde{\Psi}_*(r,s,\alpha)\geq(\bar{r}-r)^t\nabla\tilde{f}_{0*}(r,\alpha)\eta_0(\bar{r},r)-\sum_{i=1}^{m}s_i\tilde{f}_{i*}(r,\alpha)$$

$$\geq-\sum_{i=1}^{m}s_i(\bar{r}-r)^t\nabla\tilde{f}_{i_*}(r,\alpha)\eta_0(\bar{r},r)-\sum_{i=1}^{m}s_i\tilde{f}_{i_*}(r,\alpha)\ (\text{As}\ (r,s)\in Y)$$

$$\geq-\sum_{i=1}^{m}s_i\left(\tilde{f}_{i_*}(r,\alpha)-\tilde{f}_{i_*}(\bar{r},\alpha)\right)\frac{\eta_0(\bar{r},r)}{\eta_i(\bar{r},r)}-\sum_{i=1}^{m}s_i\tilde{f}_{i_*}(r,\alpha)$$

$$= -\sum_{i=1}^{m} s_i \tilde{f}_i(\bar{r}, \alpha) \text{ [As } \eta_0(\bar{r}, r) = \eta_i(\bar{r}, r)]$$

$$\geq 0 \text{ [As } s_i \geq 0 \text{ and } \bar{r} \text{ satisfies HCFP]}$$

$$\Rightarrow \tilde{\Psi}_*(\bar{r}, \bar{s}, \alpha) \geq \tilde{\Psi}_*(r, s, \alpha)$$

Similarly, $\tilde{\Psi}^*(\bar{r}, \bar{s}, \alpha) \geq \tilde{\Psi}^*(r, s, \alpha)$

Therefore, $(\bar{r}, \bar{s})$ satisfies DCFP.

As $\sum_{i=1}^{m} \bar{s}_i \tilde{f}_i(\bar{r}) = 0$, so $\tilde{\Psi}(\bar{r}, \bar{s}) = \tilde{f}_0(\bar{r})$.

Thus the two extremas are equal.

Theorem 5.3 (Strict converse duality)

Let $\tilde{f}_i(r), i = 1, 2, \ldots, m$ be H-convex differentiable fuzzy mappings on M. Let $\bar{r}$ be the solution of HCFP and let $\tilde{f}_i(r)$ satisfy the optimality criteria (4.3). If $(\hat{r}, \hat{s})$ is a solution of DHCFP and if $\Psi(r, \hat{s})$ is strictly H-convex at $\hat{r}$, then $\hat{r} = \bar{r}$ and the two extremas are equal, i.e., $\tilde{f}_0(\bar{r}) = \tilde{\Psi}(\hat{r}, \hat{s})$

Proof: We prove the theorem by the method of contradiction. Suppose that $\hat{r} \neq \bar{r}$. Given that $\bar{r}$ is a solution of HCFP. Thus, from Theorem 5.2 we conclude that there exists $\bar{s} \in ¡^m$ such that $(\bar{r}, \bar{s})$ is a solution of DHCFP.

Thus, $\tilde{\Psi}(\bar{r}, \bar{s}) = \max_{(r,s) \in Y} \tilde{\Psi}(r, s)$,

where $(\bar{r}, \bar{s}) \in Y = \{(r, s) \mid r \in M \subseteq R^n, s \in S \subseteq ¡^m, \nabla \tilde{\Psi}(r, s) = 0, s \geq 0\}$

Furthermore, let $(\hat{r}, \hat{s})$ be a solution DHCFP. Therefore,

$$\tilde{\Psi}(\hat{r}, \hat{s}) = \max_{(r,s) \in Y} \tilde{\Psi}(r, s)$$

Thus $$\tilde{\Psi}(\bar{r}, \bar{s}) = \tilde{\Psi}(\hat{r}, \hat{s}) \tag{5.8}$$

As $$(\bar{r}, \bar{s}) \in Y, \nabla \tilde{\Psi}(\hat{r}, \hat{s}) = \nabla \tilde{\Psi}(\bar{r}, \bar{s}) = 0 \tag{5.9}$$

Hence by the strict H-convexity of $\tilde{\Psi}(r, \hat{s})$ at $\hat{r}$, we have

$$\tilde{\Psi}(\bar{r}, \hat{s}) - \tilde{\Psi}(\hat{r}, \hat{s}) \text{ f } (\bar{r} - \hat{r})^t \nabla \tilde{\Psi}(\hat{r}, \hat{s}) \eta_0(\bar{r}, \hat{r}) = 0 \quad \text{(by(5.9))}$$

$$\Rightarrow \tilde{\Psi}(\hat{r}, \hat{s}) \text{ p } \tilde{\Psi}(\bar{r}, \hat{s})$$

$$\Rightarrow \tilde{\Psi}(\bar{r}, \bar{s}) \text{ p } \tilde{\Psi}(\bar{r}, \hat{s}) \quad \text{(by (5.8))}$$

$$\Rightarrow \tilde{f}_0(\bar{r}) + \sum_{i=1}^{m} \bar{s}_i \tilde{f}_i(\bar{r}) \text{ p } \tilde{f}_0(\bar{r}) + \sum_{i=1}^{m} \hat{s}_i \tilde{f}_i(\bar{r})$$

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$$\Rightarrow \sum_{i=1}^{m} \bar{s}_i \tilde{f}_i(\bar{r}) \prec \sum_{i=1}^{m} \hat{s}_i \tilde{f}_i(\bar{r})$$

$$\Rightarrow \sum_{i=1}^{m} \hat{s}_i \tilde{f}_i(\bar{r}) \succ 0, \text{ (by (4.3))},$$

which is a contradiction as $\hat{s} \geq 0$, and $\sum_{i=1}^{m} \tilde{f}_i(\bar{r}) \succ 0$. So, $\hat{r} = \bar{r}$.

Furthermore, $\tilde{\Psi}(\bar{r}, \bar{s}) = \tilde{f}_0(\bar{r}) + \sum_{i=1}^{m} \bar{s}_i \tilde{f}_i(\bar{r}) = \tilde{f}_0(\bar{r})$.

$$\Rightarrow \tilde{\Psi}(\bar{r}, \bar{s}) = \tilde{\Psi}(\hat{r}, \hat{s}) = \tilde{f}_0(\bar{r}).$$

Symmetric Duality

Symmetric duality for a set of primal and dual problems manifest that the dual of the dual is the original primal problem. Here we have introduced a different pair of symmetric dual nonlinear programming problem with the assumption of H-convexity (H-concavity) concept and established the duality results.

(PS) Minimize $\tilde{f}(p, q)$

subject to

$$\nabla_q \tilde{f}(p, q) \preceq 0 \quad (5.10)$$

$$q^T \nabla_q \tilde{f}(p, q) \succeq 0 \quad (5.11)$$

$$p \geq 0 \quad (5.12)$$

(DS) Maximize $\tilde{f}(r, s)$

subject to

$$\nabla_r \tilde{f}(r, s) \succeq 0 \quad (5.13)$$

$$r^T \nabla_r \tilde{f}(r, s) \preceq 0 \quad (5.14)$$

$$s \geq 0 \quad (5.15)$$

We assume that $\tilde{f}$ is a twice differentiable fuzzy valued function of p and q, where $p \in M \subseteq \mathbb{R}^n, q \in S \subseteq \mathbb{R}^m$. Furthermore, $\tilde{f}$ is H-convex in p for fixed q and H-concave in q for fixed p. Let $\nabla_p \tilde{f}$ and $\nabla_q \tilde{f}$ denote the gradient vectors with respect to p and q respectively.

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Theorem 5.4 Let $\tilde{f}(p,q)$ be H-convex for fixed q with $\eta(p,r)>0$ and $\tilde{f}(p,q)$ be H-concave for fixed p with $\eta(s,q)>0$. Let (p, q) be a feasible solution for (PS) and (r,s) be a feasible solution for (DS). Then $\tilde{f}(p,q)\succeq\tilde{f}(r,s)$.

Proof: From (5.16) and(5.18) we have

$$(p-r)^t\nabla_r\tilde{f}(r,s)\succeq 0$$

$\tilde{f}(p,q)$ is H-convex for fixed q with $\eta(p,r)>0$.

$$\tilde{f}(p,s)-\tilde{f}(r,s)\succeq\eta(p,r)\nabla_r\tilde{f}(r,s)(p-r)^t\succeq 0$$

$$\Rightarrow\tilde{f}(p,s)\succeq\tilde{f}(r,s) \tag{5.16}$$

Again from (5.10),(5.11) and (5.15) we have

$$(s-q)^t\nabla\tilde{f}(p,q)\preceq 0$$

$\tilde{f}(p,q)$ is H-concave for fixed p and $\eta(s,q)>0$

$$\tilde{f}(p,s)-\tilde{f}(p,q)\preceq\eta(s,q)\nabla_q\tilde{f}(p,q)(s-q)^t\preceq 0$$

$$\Rightarrow\tilde{f}(p,s)\preceq\tilde{f}(p,q) \tag{5.17}$$

From (5.16) and (5.17) we have

$$\tilde{f}(r,s)\preceq\tilde{f}(p,q).$$

5. Sufficient conditions for minmax problem

Consider the minmax problem.

$$\left.\begin{array}{l} \min\ \tilde{\Phi}(r,s) \\ \text{subject to } \tilde{f}_i(r)\preceq 0 \end{array}\right\} \tag{6.1}$$

where $\tilde{\Phi}:\mathbb{R}^n\times\mathbb{R}^m\rightarrow\tilde{F}_0,\tilde{f}_i:\mathbb{R}^n\rightarrow\tilde{F}_0, r\in M\subseteq\mathbb{R}^n,\ s\in S\subseteq\mathbb{R}^m, i=1,2,...,m.$

For $r\in M$, let $\hat{S}(r)=\left\{s\in S\mid\tilde{\Phi}(r,s)=\sup_{t\in S}\tilde{\Phi}(r,t)\right\}$ (6.2)

$\hat{S}$ is assumed to be compact and nonempty.

Theorem 6.1 Let $r^*\in M$ and $\tilde{f}_i(\cdot)$ is a component wise differentiable H-convex fuzzy mappings of r for i = 1,2,….p, as well as $\tilde{\Phi}(.,s)$ be a differentiable H-convex fuzzy mapping of r for every $s\in S$. If there exists a positive integer $\beta, 1\leq\beta\leq n+1$ and scalars

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$\lambda_i \geq 0,\ i = 1,2,....\beta$ such that $\sum_{i=1}^{\beta} \lambda_i \neq 0$ and scalars $\mu_i \geq 0,\ i = 1,2,....,p$ and if $\exists$ vectors $s^i \in \hat{S}(r^*), i = 1,2,...,\beta$ such that

$$\sum_{i=1}^{\beta} \lambda_i \nabla_r \tilde{\Psi}(r^*, s^i) + \sum_{i=1}^{p} \mu_i \nabla_r \tilde{f}_i(r^*) = 0 \tag{6.3}$$

$$\mu_i \tilde{f}_i(r^*) = 0,\ i = 1,2,...,p \tag{6.4}$$

then r^* is a minmax solution of (6.1).

Proof: Let on the contrary r^* be not a minmax solution but the conditions of the theorem are satisfied. Then there exists $r^0 \in M$ such that

$$\sup_{s \in S} \tilde{\Psi}(r^0, s) < \sup_{s \in S} \tilde{\Psi}(r^*, s) \tag{6.5}$$

As $\tilde{f}_i(r^0) \leq 0$ and $\mu_i \geq 0$, thus for $i = 1,2,....,p$

$$\mu_i \tilde{f}_i(r^0) \leq 0 \tag{6.6}$$

But for $i = 1,2,....,\beta$

$$\sup_{s \in S} \tilde{\Psi}(r^*, s) = \tilde{\Psi}(r^*, s^i) \tag{6.7}$$

and
$$\sup_{s \in S} \tilde{\Psi}(r^0, s) = \tilde{\Psi}(r^0, s^i) \tag{6.8}$$

From (6.5), (6.7) and (6.8) we have

$$\tilde{\Psi}(r^0, s^i) < \tilde{\Psi}(r^*, s^i) \tag{6.9}$$

$$\Rightarrow \sum_{i=1}^{\beta} \lambda_i \tilde{\Psi}(r^0, s^i) + \sum_{i=1}^{p} \mu_i \tilde{f}_i(r^0) < \sum_{i=1}^{\beta} \lambda_i \tilde{\Psi}(r^*, s^i) + \sum_{i=1}^{p} \mu_i \tilde{f}_i(r^*) \tag{6.10}$$

From the H-convexity assumptions of $\tilde{\Psi}$ and $\tilde{f}_i$ we have

$$\tilde{\Psi}(r^0, s^i) - \tilde{\Psi}(r^*, s^i) \succeq (r^0 - r^*)^t \nabla_r \tilde{\Psi}(r^*, s^i)\, \eta_0(r^0, r^*)$$

and
$$\tilde{f}_i(r^0) - \tilde{f}_i(r^*) \succeq (r^0 - r^*)^t \nabla_r \tilde{f}_i(r^*)\, \eta_i(r^0, r^*), i = 1,2,....,p$$

i.e.,
$$\tilde{\Psi}_*(r^0, s^i, \alpha) - \tilde{\Psi}_*(r^*, s^i, \alpha) \geq (r^0 - r^*)^t \nabla_r \tilde{\Psi}_*(r^*, s^i, \alpha)\, \eta_0(r^0, r^*) \tag{6.11}$$

$$\tilde{\Psi}^*(r^0, s^i, \alpha) - \tilde{\Psi}^*(r^*, s^i, \alpha) \geq (r^0 - r^*)^t \nabla_r \tilde{\Psi}^*(r^*, s^i, \alpha)\, \eta_0(r^0, r^*) \tag{6.12}$$

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and $$\tilde{f}_{i_*}\left(r^0,\alpha\right)-\tilde{f}_{i_*}\left(r^*,\alpha\right)\geq\left(r^0-r^*\right)^t\nabla_r\tilde{f}_{i_*}\left(r^*,\alpha\right)\eta_i\left(r^0,r^*\right), i=1,2,....,p \tag{6.13}$$

$$\tilde{f}_i^*\left(r^0,\alpha\right)-\tilde{f}_i^*\left(r^*,\alpha\right)\geq\left(r^0-r^*\right)^t\nabla_r\tilde{f}_i^*\left(r^*,\alpha\right)\eta_i\left(r^0,r^*\right), i=1,2,....,p \tag{6.14}$$

Now by using (6.11) and (6.13) we get

$$\sum_{i=1}^{\beta}\lambda_i\tilde{\Psi}_*\left(r^0,s^i,\alpha\right)-\sum_{i=1}^{\beta}\lambda_i\tilde{\Psi}_*\left(r^*,s^i,\alpha\right)\geq\sum_{i=1}^{\beta}\lambda_i\left(r^0-r^*\right)^t\nabla_r\tilde{\Psi}_*\left(r^*,s^i,\alpha\right)\eta_0\left(r^0,r^*\right) \tag{6.15}$$

and $$\sum_{i=1}^{p}\mu_i\tilde{f}_{i_*}\left(r^0,\alpha\right)-\sum_{i=1}^{p}\mu_i\tilde{f}_{i_*}\left(r^*,\alpha\right)\geq\sum_{i=1}^{p}\mu_i\left(r^0-r^*\right)^t\nabla_r\tilde{f}_{i_*}\left(r^*,\alpha\right)\eta_i\left(r^0,r^*\right) \tag{6.16}$$

As the above two equations (6.15) and (6.16) are true for $\eta_i\left(r^0,r^*\right)>0$ and $\eta_i\left(r^0,r^*\right)>0$, hence both will be true for $\eta_i\left(r^0,r^*\right)\geq 1$ and $\eta_i\left(r^0,r^*\right)\geq 1$. Therefore, we can write

$$\sum_{i=1}^{\beta}\lambda_i\tilde{\Psi}_*\left(r^0,s^i,\alpha\right)-\sum_{i=1}^{\beta}\lambda_i\tilde{\Psi}_*\left(r^*,s^i,\alpha\right)\geq\sum_{i=1}^{\beta}\lambda_i\left(r^0-r^*\right)^t\nabla_r\tilde{\Psi}\left(r^*,s^i,\alpha\right) \tag{6.17}$$

and $$\sum_{i=1}^{p}\mu_i\tilde{f}_{i_*}\left(r^0,\alpha\right)-\sum_{i=1}^{p}\mu_i\tilde{f}_{i_*}\left(r^*,\alpha\right)\geq\sum_{i=1}^{p}\mu_i\left(r^0-r^*\right)^t\nabla_r\tilde{f}_{i_*}\left(r^*,\alpha\right) \tag{6.18}$$

Now from (6.17) and (6.18) we have

$$\sum_{i=1}^{\beta}\lambda_i\tilde{\Psi}_*\left(r^0,s^i,\alpha\right)+\sum_{i=1}^{p}\mu_i\tilde{f}_{i_*}\left(r^0,\alpha\right)-\sum_{i=1}^{\beta}\lambda_i\tilde{\Psi}_*\left(r^*,s^i,\alpha\right)-\sum_{i=1}^{p}\mu_i\tilde{f}_{i_*}\left(r^*,\alpha\right)$$

$$\geq\left(r^0-r^*\right)^t\left[\sum_{i=1}^{\beta}\lambda_i\nabla_r\tilde{\Psi}_*\left(r^*,s^i,\alpha\right)+\sum_{i=1}^{p}\mu_i\nabla_r\tilde{f}_{i_*}\left(r^*,\alpha\right)\right]$$

$= 0$ (by (6.3))

$$\Rightarrow\sum_{i=1}^{\beta}\lambda_i\tilde{\Psi}_*\left(r^0,s^i,\alpha\right)+\sum_{i=1}^{p}\mu_i\tilde{f}_{i_*}\left(r^0,\alpha\right)\geq\sum_{i=1}^{\beta}\lambda_i\tilde{\Psi}_*\left(r^*,s^i,\alpha\right)-\sum_{i=1}^{p}\mu_i\tilde{f}_{i_*}\left(r^*,\alpha\right),$$

which contradicts (6.10). Hence r^* is a minmax solution.

Example 6.1 We have considered the following example to illustrate Theorem 6.1.

Let us defined two fuzzy mappings $\tilde{f}(r,s)$ and $\tilde{g}(r)$ whose α -level sets are given by

$$\tilde{f}(r,s)[\alpha]=\left[\left(\frac{2\alpha-4}{(1+rs)^2},\frac{-(1+\alpha)}{(1+rs)^2},\alpha\right)\bigg|\, 0\le\alpha\le1\right]$$

$$\tilde{g}(r)[\alpha]=\left[\left((4-2\alpha)\left(\frac{1}{2-r^2}-1\right),(1+\alpha)\left(\frac{1}{2-r^2}-1\right),\alpha\right)\bigg|\, 0\le\alpha\le1\right],$$

where, $-1\le r\le1$ and $1\le s\le2$.

Let $\eta_0\left(r^0,r^*\right)=\frac{\left(1+r^*s^i\right)^2}{\left(1+r^0s^i\right)^2}$, $\eta_1\left(r^0,r^*\right)=\frac{\left(2-r^{*2}\right)}{\left(2-r^{02}\right)}$ and

$$\hat{S}(r)=\begin{cases}\{2\} & \text{if } -1\le r<-2/3\\ \{1,2\} & \text{if } r=-2/3\\ \{1\} & \text{if } -2/3<r<0\\ S & \text{if } r=0\\ \{2\} & \text{if } 0<r\le1\end{cases}$$

Let $S=\{s\,|\,1\le s\le2\}$

Thus, $\tilde{f}(r,s)$ and $\tilde{g}(r)$ are differentiable H-convex fuzzy mappings with respect to η_0 and η_i respectively, which can be easily verified.

From the equation (6.3) we get the following results.

For $r=-1, s^1=2$ and $\mu_1=0$, we have $\lambda_1=0$

For $-1<r<-2/3, s^1=2$ and $\mu_1=0$, we get $\lambda_1=0$.

For $r=-2/3, s^1=2, s^2=2$ and $\mu_1=0$, we have $\lambda_1=2\lambda_2$

For $-2/3<r<0, s^1=1$ and $\mu_1=0$, we get $\lambda_1=0$.

For $r=0$ and $s\in S$, we have $\lambda_1=0$

For $0<r\le1, s^1=2$ and $\mu_1=0$, we get $\lambda_1=0$.

So, for $r = -2/3, \mu_1 = 0, p = 1, \beta = 2, \lambda_1 = 2, \lambda_2 = 1, s^1 = 1$ and $s^2 = 2$ the sufficient conditions of the minmax theorem is satisfied and for all other admissible values of r, the conditions of the theorem is not satisfied with the elements of $\hat{S}$.

Hence $r = -2/3$ is a minmax solution.

7. Discussion

The problem of Harmonic convex fuzzy programming (HCFP) was developed by taking main function and collection of limitations as differentiable H-convex fuzzy mappings. Various fuzzy programming models were addressed in, in which the fuzzy mappings are specific generalized convexities. As we discussed in Remark 3.1 that a convex H-convex fuzzy mapping can be convex fuzzy mapping by choosing the correct value of π(r1, r2), but the converse is not necessarily valid, implying that convex fuzzy mapping may not be H-convex.HCFP is thus a problem of optimization that is different from the convex fuzzy programming problem (FCP) described in and can be addressed via HCFP not only H-convex fuzzy mappings but also most generalized convex fuzzy mappings such as convex, invex and B-vex fuzzy mappings. The results of duality such as weak duality, forward duality, strict converse duality and the results of symmetric duality are obtained for HCFP problem which is different from the works in. Minmax problem has been studied by in a real field and has studied minmax theorems without the principle of differentiability under the fuzzy setting.We have currently studied the necessary optimality conditions of a minmax problem involving main function and collection of limitations as differentiable H-convex fuzzy mapping which is different from the previous works. In addition, H-convex fuzzy mapping may be the initial work for generalized H-convex fuzzy mappings, as it can be extended to H-invex and H-univex fuzzy mappings again to discuss some more extended convex and harmonic convex fuzzy mappings.

8. Conclusion and futureresearch

With the H-convex (H-concave) fuzzy map- pings, the differentiability principle has been applied and some interesting results have been obtained that connect H-convex (H-concave) fuzzy mappings to some known generalized convexities such as pseudoconvex (pseudoconcave) and quasiconvex (quasiconcave) fuzzy mappings. The global minimal properties (absolute maximum) of H-convex (H-concave) fuzzy mapping have been discussed. Sufficient conditions of optimality, conditions of KKT, results of duality and minmax problem were studied by supposing both the main function and set of limitations as differentiable H-convex fuzzy mappings. The findings were demonstrated with the correct examples. Harmonic convexity is the generalized convexity which can be derived from such convexities as convexity, invexity and B- vexity. We plan to use generalized H-convex fuzzy mappings to research the problems of multi-objective optimization, fractional programming issues and some fundamental inequalities in our future work.

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