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A Probabilistic Approach in Modeling the Growth of Cell Population

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Abstract:
A number of contraction mappings have been drawn by several researchers to generalize Banach type contraction. Some Cauchy sequences are applied to show the uniqueness of the fixed points obtained on the basis of certain fixed point theorems on these mappings. The aforementioned technique, using these theorems, is extensively used in finding the solution of numerous mathematical quandaries. M. Rotenberg, in his approach to model the growth of cell population under certain boundary conditions, utilized fixed point theorem in general Banach space. He suggested a mathematical model, represented by the partial differential equation shown in equation (1), by considering the population density of cells as a function of two parameters (degree of maturity, μ and maturation velocity, v) and time.
In Rotenberg mathematical model, application of fixed point method is found consistent by considering the quandary as a non linear boundary value problem because, each cell is assorted by its degree of maturity and velocity of maturation which are fixed and coupled biological boundaries as $\mu = 0$ and $v = c$ (a positive value). Further, due to concentration and other density depending effects, the transition rates under a nutrient environment may fluctuate, so that, it must be a function of population density. This is again the clear depiction of the problem as non linear which is most suitable for modeling by using the said procedure and hence, applicable for employing a fixed point theorem.
Our approach in this paper is to further investigate the result by applying some more contraction mapping with associated fixed point theorem in probabilistic metric space.

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Introduction

The growth of cell population is represented by a specific mathematical model introduced by M. Rotenberg [4] in which each cell is categorized by its degree of maturity μ and velocity of maturation $v = \frac{d\mu}{dt}$. The density of the cell population, represented by the function $\Psi(t, \mu, v)$, is expressed in the form of following partial differential equation.

$$\frac{\partial \Psi}{\partial t}(t, \mu, v) = -v \frac{\partial \Psi}{\partial t}(t, \mu, v) - \sigma(\mu, v)\Psi(t, \mu, v) + \int_0^c r(\mu, v, v')\Psi(t, \mu, v')dv' \qquad (1)$$

Where μ∈[0,a]; v, v'∈[0,c] with a>0,c>0. The degree of maturity at birth is μ = 0 and the same at mitosis is μ = c. Meaning there by, the cells are born at μ = 0 and divide at μ = c. The kernel r (μ,v,v') is termed as

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the transition rate. This stipulates the transition of cells from velocity of maturation v' to v, whereas total transition cross-section is denoted by σ (μ,v).
The numerical solution of the model represented by (1) is obtained by Rotenberg by using Fokker-Plank approximation [5], while analytical solution is found by Vaneder Mee and Zweifel [5] for various boundary conditions employing the technique of Eigen function. Lebowitz and Rubinow's boundary value conditions [6] were used to entrench that the Cauchy problem assigned with equation (1), which is subdued by a positive C_0 semi group. A. Dehici, A. Jeribi and K. Latrach [14], in their spectral analysis of (1), pointed out the general linear transition rule between mother and daughter cells at mitosis. This supports the previous notions.

It was established by Rotenberg that the cells concerned are under a nutrient environment which may fluctuate due to concentration and other effects depending on density. Therefore, the transition rates σ and r must be the function of the population density. Hence, the problem modeled by (1), is non linear and biological constraints at μ = 0 and μ = c are fixed. These observations led K. Latrach and A. Jeribi [8] to improve the Rotenberg model assuming the transition rate and total cross-section dependent on the population density. Thus, the non linear formulation between daughter and mother cells at mitosis depicts the boundary conditions. Hence, the problem concerned is given as

$$v\frac{\partial \Psi}{\partial \mu}(\mu, v) + \lambda\Psi(\mu, v) + \sigma\big(\mu, v, \Psi(\mu, v)\big) = \int_0^c r(\mu, v, v')\Psi(\mu, v')dv' \qquad (2)$$

Here, σ and r are non linear functions of Ψ and λ where, λ is a complex number.
Further, the boundary conditions are modeled as

$$\Psi|\Gamma_0 = K(\Psi|\Gamma_1) \qquad (3)$$

Where, $\Gamma_0 = \{0\} \times [0, c]$ and $\Gamma_0 = \{0\} \times [0, c]$, $\Psi|\Gamma_0$ (resp. $\Psi|\Gamma_1$) depicts the constraint of Ψ to Γ_0 (resp. $\Psi|\Gamma_1$). Here, K is a non linear operator.

A number of theorems regarding equation (2) along with boundary conditions (3) have been given in L_p spaces with $p \in (1, \infty)$. The analysis of stationary transport equation in kinetic theory of gas molecules with solid walls bounded by the region, where the flow of gas is described, has earlier been shown by the primary author of references [7][9]. Contrary to biological model, the boundary conditions are taken as linear. Topological arguments using compactness results for $p \in (1, \infty)$ has been taken in to account during the analysis. Due to lack of compactness, the problem given by (2) and (3) establishing the existence results as shown in references [7][8][9], remained open in L_1 spaces for both population dynamics and kinetic theory of gas.
The main purpose of this paper is to examine the establishment of the existed result further by using certain fixed point theorems on contraction mappings in probabilistic metric space along with the properties of weakly compact sets in the growth of cell population, which may be used as the technique in environmental pollution protection and other cell population growth related parameters.

Main Approach of Rotenberg

Rotenberg has established in his paper that the model of the growth of cell population is nonlinear. A nonlinear weakly compact operator is introduced in obtaining the proof of the model by transforming it into a fixed point problem. In other words, it is the transformation of the bounded sets into the weakly compact sets. Schauder fixed point theorem [2] is used in his approach to find at least one fixed point, that is, to find at least one solution of the model of the growth of cell population as given by (1). It is manifested that, the concept of general metric space

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is employed in finding the solution of the model of the growth of cell population in the concerned approach.

Preliminaries

Before approaching the main result it is necessary to deal with some basics required in drawing out the desired conclusion.

Definition 1: [1] A mapping $f: X \to X$ on a metric space (X, d) is called Banach contraction if for a contraction constant $k \in [0,1)$, the following condition holds good.

$$d(\mathrm{f}_x, \mathrm{f}_y) \leq kd(x, y), \text{ for all } x, y \in X.$$

Definition 2: [12] A sequence $< \alpha_n >$ in a metric space (X, d) is said to converge at $\alpha_0 \in \mathrm{X}$ if $d(\alpha_n, \alpha_0) \to 0$ when $n \to \infty$ or if for a number $N(\varepsilon)$, we have

$$d(\alpha_n, \alpha_0) < \varepsilon.$$

Definition 3: [12] A sequence $< \alpha_n >$ in a metric space (X, d) is a Cauchy sequence if and only if for every small positive number ε, there exists a positive integer $N(\varepsilon)$ such that

$$m, n \geq N(\varepsilon) \Rightarrow d(\alpha_m, \alpha_n) < \varepsilon.$$

Definition 4: [12] Completeness of a metric space is defined, if every Cauchy sequence in the said metric space converges to a point which lies in it. Otherwise, it is said to be incomplete.

Definition 5: [12] A Contraction mapping of a metric space (X, d) is defined as a mapping $f: X \to X$ for any real number $k \in [0,1)$ such that $d(f(\mathrm{x}), f(\mathrm{y})) \leq kd(x, y)$, for all $x, y \in X$, where k is a contraction constant.

Note: It may effortlessly be shown that these mappings are uniformly continuous. We simply mean continuous by a uniformly continuous mapping.

Definition 6: [12] Consider a mapping $f: X \to X$ on a non empty set X. A point α_0 is called a fixed point in X if

$$f(\alpha_0) = \alpha_0, \text{ for every } \alpha_0 \in X.$$

Theorem 1: [12] Every convergent sequence $< \alpha_n >$ in a metric space (X, d) is a Cauchy sequence.

Theorem 2: [12] Consider a contraction mapping $f: X \to X$ defined on a non empty, complete metric space (X, d). Then there exists a unique fixed point of the mapping f in X. Also, the iterative sequences $\alpha, f(\alpha), f\big(f(\alpha)\big)$, and so on, converge to some fixed points of f in X, for $\alpha \in X$.

Definition 7: [13] A non decreasing, left continuous mapping $F: R \to \mathrm{R}^+$ is defined as a distribution function if, for $r \in R$, inf $F(\mathrm{r}) = 0$ and sup $F(\mathrm{r}) = 1$.

Now, let F^+ be the collection of all distribution functions F such that $F(0) = 0$ and F is a non decreasing, left continuous mapping $F: R \to [0,1]$ with sup $F(\mathrm{r}) = 1$.

Definition 8: [13] We define a probabilistic metric space (in short, PM-space) as an ordered pair (H, F) for a non empty set H and $F: H \times H \to \mathrm{F}^+$ where $F(p, q)$ is represented by $F_{p,q}$ for each $(p, q) \in H \times H$ satisfying the conditions:

(i) $F_{x,y}(\mathrm{r}) = 1$, for all $r > 0$, iff $x = y;\ x, y \in H$

(ii) $F_{x,y}(0) = 0$, for all $x, y \in H$

(iii) $F_{x,y}(\mathrm{r}) = F_{y,x}(\mathrm{r})$, for all $x, y \in H$ and $r \epsilon R$

(iv) $F_{x,y}(\mathrm{r}) = 1$ and $F_{y,z}(\mathrm{s}) = 1$, then $F_{x,z}(\mathrm{r} + \mathrm{s}) = 1$, for all $x, y, z \in H$ and $r, s \in R$.

The pair (H, F) is called probabilistic semi-metric space if only (i) and (iii) hold good.

Definition 9: [13] A triangular norm denoted by t-norm is a mapping $t: [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(i) $\mathrm{t}(\alpha, 1) = \alpha$, for every $\alpha \in [0,1]$

(ii) $\mathrm{t}(\alpha, \beta) = \mathrm{t}(\beta, \alpha)$, for every $\alpha, \beta \in [0,1]$

(iii) $\mathrm{t}(\alpha, \gamma) = \mathrm{t}(\beta, \delta)$, if $\alpha \geq \beta, \gamma \geq \delta$

(iv) $\mathrm{t}\big(\alpha, \mathrm{t}(\beta, \gamma)\big) = \mathrm{t}(\mathrm{t}(\alpha, \beta), \gamma);\ \alpha, \beta, \gamma \in [0,1]$.

Definition 10: [10] The triplet (H, F, t) with pair (H, F) as probabilistic metric space and t a triangular norm or t-norm is called a Menger probabilistic metric space satisfying the inequality $F_{x,y}(\mathrm{r} + \mathrm{s}) \geq \mathrm{t}\Big(F_{x,z}(\mathrm{r}), F_{z,y}(\mathrm{s})\Big)$, for every $x, y, z \in H$ and $r > 0, s > 0$.

Definition 11: [10] Consider the Menger probabilistic metric space (H, F, t) and sup $\mathrm{t}(l, l) = 1$, $0 < l < 1$. Then,

(i) A sequence $< \alpha_n >$ in H is called t-convergent at $\alpha \in H$ if for given $\varepsilon > 0$ and $\lambda > 0$ we have a

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positive integer $N(\varepsilon,\lambda)$ such that $F_{\alpha_n,\alpha}(\varepsilon) > 1-\lambda$, whenever $n \geq N(\varepsilon,\lambda)$. And, we may write

$$\alpha_n \to \alpha.$$

(ii) The sequence $< \alpha_n >$ in H is said to be a t-Cauchy sequence if for a given $\varepsilon > 0$ and $\lambda > 0$ we have a positive integer $N(\varepsilon,\lambda)$ such that $F_{\alpha_n,\alpha_m}(\varepsilon) > 1-\lambda$, whenever $m,n \geq N(\varepsilon,\lambda)$.

(iii)The Menger probabilistic metric space (H,F,t) is called t-complete if every t-Cauchy sequence is t-convergent to some limit point in H.

Definition 12: Contractive mapping under Menger PM-space [10]**:**

A mapping $f: H \to$ H is a contraction mapping in a Menger PM-space (H,F,t), if for all $x \neq y$, $x,y \in H$,

$$F_{f(x),f(y)}(r) \geq F_{x,y}(r),$$
$$\text{for every } r > 0,$$
$$F_{f(x),f(y)} \neq F_{x,y}.$$

Main Results and discussion

The distance between any two points in a general metric space is a non negative real number. The idea was extended by K. Menger [3] in 1942 by introducing the distribution function of random variables with non negative real numbers. As discussed earlier in the paper, the Rotenberg approach to model the growth of cell population is to find a solution of the said problem by transforming the bounded sets into the weakly compact sets and using Schauder fixed point theorem in general metric space assuming the two parameters as the degree of maturity, μ and velocity of maturation, v of the cells. Therefore, application of the probabilistic metric space in place of general metric space to model the growth of cell population seems more appropriate in dealing with the same as the growth of cell population is a continuous function of time and velocity of maturation.

Thus, the main purpose of this paper is to establish the existed result by making the proof of a theorem to get a unique fixed point in the probabilistic metric space which is the more general form of the Schauder fixed point theorem.

Our Approach

To prove our main result we need the following lemma and theorem.

Theorem 3: Let (H,F,t) be a complete Menger PM space and $t(x,x) \geq x \ \ \forall\, x \in [0,1]$.

If $f: H \to H$ is a continuous function and $\{p_n\}$ is a Cauchy sequence defined by $p_n = fp_{n-1}$ converges to $p \in H$. Then, p is a fixed point of f.

Proof: Since, $p_n \to p$ so

$$\lim F_{p_n p}(x/2) = 1 \ , \ x > 0$$

Due to continuity of f, we have

$$\lim F_{fp_n\, fp}(x/2) = 1 \ , \ x > 0$$

$$F_{fp\ p}(x) \geq t[F_{fpfp_n}(x/2), F_{fp_n p}(x/2)]$$

$$= t[F_{fpfp_n}(x/2), F_{p_{n+1}p}(x/2)] \text{ , for } n \to \infty$$

$$F_{fp\ p}(x) \geq t[1,1] = 1 \quad \forall \quad x > 0$$

By the property of distribution function, we have $f(p) = p$

Lemma 1: Let (H,F,t) be a Menger PM space, where t is continuous. And, $\exists\ k > 1$ such that

$$F_{fpf^2p}(kx) \leq F_{pfp}(x) \ , \ x > 0$$

Suppose $f: H \to H$ is onto mapping then $\exists$ a Cauchy sequence in H.

Proof: Since, f is onto so, for $p_0 \in H$, $\exists\ p_1 \in H$ such that $f(p_1) = p_0$. Now, construct a sequence $\{p_n\}$ as $f(p_n) = p_{n-1}$, $n = 1, 2, 3, \ldots$. Then,

$$F_{fp_n f^2 p_n}(kx) \leq F_{p_n fp_n}(x)$$

$$F_{p_{n-1}p_{n-2}}(kx) \leq F_{p_n p_{n-1}}(x)$$

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$$F_{p_n p_{n-1}}\left(\frac{1}{k}x\right) \geq F_{p_{n-1}p_{n-2}}(x)$$

$$F_{p_n p_{n-1}}(k'x) \geq F_{p_{n-1}p_{n-2}}(x)$$

Since, $\frac{1}{k} = k' \in (0,1)$

Therefore, $\{p_n\}$ is a Cauchy sequence.

Theorem 4: Let (H, F, t) be a complete Menger space and $f : H \to H$ is continuous and onto mapping satisfying the property of lemma. Then, f has a fixed point.

Proof: By above lemma $\exists$ a Cauchy sequence $\{p_n\}$ in H. Since, H is complete so, $p_n \to p \in X$. Then, by theorem 3, p is a fixed point of f.

Theorem 5: Let (H, F, t) be a complete Menger probabilistic metric space where $F_{p,q}$ is strictly increasing distribution function and $f : H \to H$ is continuous onto mapping. If $\exists\, k > 1$ such that

$$F_{fpfq}(kx) \leq \min\{F_{pfp}(x), F_{qfq}(x), F_{pq}(x)\}$$

Then, f has a fixed point.

Proof: Let $p_0 \in X$, as done in lemma 1, we can construct a sequence as, $p_{n-1} = f(p_n)$, $n = 1, 2....$

$$F_{p_{n-1}p_n}(kx) = F_{fp_n\, fp_{n+1}}(kx) \leq \min\{F_{p_n p_{n-1}}(x), F_{p_{n+1}p_n}(x), F_{p_n p_{n+1}}(x)\}$$

$$F_{p_{n-1}p_n}(kx) \leq \min\{F_{p_n p_{n-1}}(x), F_{p_n p_{n+1}}(x)\}$$

Since, F is non-decreasing function and $k > 1$, so $k\,x > x$. Then,

$$F_{p_{n-1}p_n}(kx) \leq F_{P_n p_{n+1}}(x) \quad \text{i.e}$$

$$F_{p_n p_{n+1}}\left(\frac{1}{k}x\right) = F_{p_n p_{n+1}}(k'x) \geq F_{p_{n-1}p_n}(x)\ ,\ \frac{1}{k} = k' \in (0,1)$$

So, $\{p_n\}$ is a Cauchy sequence. Since, H is complete so, $p_n \to p \in H$, then by above theorem, p is a fixed point of f.

The proof of above theorem will be the technique to find the solution of the model expressed by equation (1).

Thus, using the contraction mapping given in equation (4) and fixed point theorem as discussed above, the growth of cell population can be modeled along with its solution as given by Rotenberg. This may be used as the technique in establishing the existed result for cell population growth.

Conclusion

The dramatic increase in cell population of certain kind of bacteria may affect the environmental factors like water, food, or even the air which we inhale. Further, the growth of cell population can be measured during certain time interval which may also have an effect on the plant growth in positive or negative sense. This may sometimes increase toxic organics in the soil and atmosphere, becoming a cause of biodegradation. Using the contraction mapping given in equation (4) and fixed point theorem as mentioned, we can furnish the Rotenberg model under probabilistic metric space, which may have decisive advantage in modeling the growth of

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cell population. The technique may also be helpful in the production of antibiotics or other such items at greater extent in the pharmaceutical industry as well. Further, this modeling technique may essentially be employed in environmental pollution protection and other cell population growth related parameters.

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REFERENCES

1. Banach, S. (1922), Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrals. Fundam. Math. 3, 133-181.
2. Schauder, J. (1930), Der fixpunktraz functional saumen, Studia Math. 2, 171 – 180.
3. Menger, K. (1942), Statistical metric, Proc. Nat. Acad. (USA) 28, 535.
4. Rotenberg, M. (1983), Transport theory for growing cell populations, J. Theo. Biol. 103:181-199.
5. Van der Mee, C. and Zweifel, P. (1987), A Fokker-Planck equation for growing cell populations J. Math. Biol. 25: 61-72.
6. Greenberg, W., Van der Mee, C. and Protopopescu, V. (Birkhauser, 1987), Boundary Value Problems in Abstract Kinetic Theory.
7. Latrach, K. (1996), On a nonlinear stationary problem arising in transport theory, J. Math. Phys. 37: 1336-1348.
8. Latrach, K. and Jeribi, A. (1999), A nonlinear boundary value problem arising in growing cell populations, Nonlinear Anal. T.M.A. 36: 843-862.
9. Latrach, K. (2001), Compactness results for transport equations and applications, Math. Models Methods Appl. Sci.11: 1181-1202.
10. Dehici, A. Jeribi, A. and K. Latrach (2002), On a transport operator arising in growing cell populations I. Spectral analysis, Advances in Math. Research, (ed.) G. Oyibo, Nova Sci. Publishers, pp. 159-186.
11. Razani, A. (2006), New Zealand Journal of Mathematics, Vol. 35, 109-114.
12. Latracha, K., et. al. (2006),Some fixed point theorems of the Schauder and the Krasnosel'skii type and application to nonlinear transport equations J. Differential Equations 221 : 256 – 271
13. Peeler, Luke (2011), Metric Spaces and the contraction mapping principle, Math. Unchicago, edu.
14. Tripathi, Piyush Kumar, et.al. (2016), Global Journal of Pure and Applied Mathematics, Volume 12, Number 2, pp. 1629-1634.
15. Sharma, J.N. (1979), Topology, 29th edn., Krishna Prakshan Media (P) Ltd., Meerut, pp. 77-148.