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Means of Certain Arithmetic Functions

G Sudhaamsh Mohan Reddy[1], S Srinivas Rau[2], B Uma[3]
[1& 2] Faculty of Science and Technology, Icfai Foundation for Higher Education, Hyderabad-501203, INDIA, *dr.sudhamshreddy@gmail.com, rauindia@yahoo.co.in*
[3]CTW, Military College, SECUNDRERABAD-500015,INDIA,u*bidarahalli699@gmail.com*

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Abstract:
Means of $\{\frac{\mu^2(n)\sigma(n)}{n}\}$,$\{\frac{\sigma(p)}{p}\}$ and $\{\sum_{p|n}\frac{1}{p}\}$ are computed to be $\frac{\zeta(2)}{\zeta(4)}$, 0 and $\sum\frac{1}{p^2}$ respectively. An estimate for $\sum\frac{1}{p^2}$ is given.As a corollary we have the upper bound $\frac{25}{16}$ for the mean value of L-series values $\{\frac{L_d(1)}{L_d(2)}\}_{d\ squarefree}$

Keywords: *measuring, Arithmetic function.*

I INTRODUCTION

Let $\sigma(n)=\sum_{k|n}k$ .Then $\frac{\sigma(n)}{n}=\sum_{k|n}\frac{1}{k}$. The mean of an arithmetic function $f(n)$ is defined to be $Lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}f(n)$.

The mean of the arithmetic function $\frac{\sigma(n)}{n}$ is known to be $\zeta(2)=\sum\frac{1}{n^2}=\frac{\pi^2}{6}$ ([1], Th 3.6). We adapt this method to compute the mean of $\{\frac{\mu^2(n)\sigma(n)}{n}\}_{n=1}^{\infty}$ the sequence of sums of reciprocals of divisors of squarefree integers : the value is $\frac{\zeta(2)}{\zeta(4)}=\sum\frac{\mu^2(n)}{n^2}$ (Prop 1 (a)). For the special choice of subsequence of primes, we find the mean to be 0 (Prop 1 (b)).The "formal guess" $\sum_p\frac{1}{p^2}$ turns out to be the mean of a different function $f(n)=\sum_{p|n}\frac{1}{p}$ (Remark 1). For this we use the Tauberian result in Hlawka et al ([3] p200). We discuss the value of $\sum_p\frac{1}{p^2}$ and an upper bound for it.

Proposition 1

1. $$Lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\frac{\mu^2(n)\sigma(n)}{n}=\frac{\zeta(2)}{\zeta(4)}=\sum_{n=1}^{\infty}\frac{\mu^2(n)}{n^2}=\frac{15}{\pi^2}$$

2. $$Lim_{x\to\infty}\frac{1}{x}\sum_{p\le x}\frac{\sigma(p)}{p}=0$$

Proof. (a)

$$\sum_{n\le x}\frac{\mu^2(n)\sigma(n)}{n}=\sum_{s\le x,ssqfree}(\sum_{d|s}\frac{1}{d})$$

$$=\sum_{d\le x,dsqfree}\frac{1}{d}(\sum_{q\le x/d,qsqfree}1)$$

$$=\sum_{d\le x,dsqfree}\frac{1}{d}\{\frac{x}{d}+\bigcirc(1)\}$$

$$=x\sum_{d\le x,dsqfree}\frac{1}{d^2}+\bigcirc(\sum_{d\le x,dsqfree}\frac{1}{d})$$

$$=x\sum_{n\le x}\frac{\mu^2(n)}{n^2}+\bigcirc(logx)$$

$$(since\sum_{d\le x}\frac{1}{d}\le\sum_{n\le x}\frac{1}{n}\approx logx([1],Th3.2))$$

Dividing by x and letting $x\to\infty$ we have the

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mean to be $\sum_n \frac{\mu^2(n)}{n^2} = \frac{\zeta(2(}{\zeta(4)}$ as claimed. Since $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ the numerical value is $\frac{15}{\pi^2}$ ([4],p 232).

(b)Define

$$f(n) = \begin{cases} 0 & \text{ifnnotprime} \\ \frac{\sigma(p)}{p} & \text{ifn} = \text{p, prime} \end{cases}$$

We compute the mean of $f(n)$:

$$\frac{1}{x}\sum_{n\leq x} f(n) = \frac{1}{x}\sum_{p\leq x} \frac{\sigma(p)}{p}$$

$$= \frac{1}{x}\sum_{p\leq x} (1+\frac{1}{p})$$

$$= \frac{1}{x}\{\sum_{p\leq x} 1 + \sum_{p\leq x} \frac{1}{p}\}$$

$$= \frac{\pi(x)}{x} + \frac{1}{x}\sum_{p\leq x} \frac{1}{p}$$

But $\sum_{p\leq x} \frac{1}{p} \approx loglogx$ ([1], Theorem 4.12). Also as $x \to \infty$, $\frac{\pi(x)}{x} \to 0$ ([4], Cor 2, p24). Hence

$$Lim_{x\to\infty} \frac{1}{x}\sum_{p\leq x} \frac{\sigma(p)}{p}$$

$$= Lim_{x\to\infty} \frac{\pi(x)}{x}$$

$$+ Lim_{x\to\infty} \frac{1}{x}(\sum_{p\leq x} \frac{1}{p}) = 0 + 0 = 0$$

Remark 1 We point out another approach to proofs of Prop1, by ([3], p200) Tauberian methods. Let $f(n)$ be a bounded arithmetic function of mean 0. Then

$$\sum_{n\leq x} \frac{f(n)}{n} = \frac{1}{x}\sum_{n\leq x} (\sum_{m|n} f(m)) + \circ(1)$$

We choose $f(n) = \frac{1}{n}$ and apply the above; letting $x \to \infty$ we have $\sum \frac{1}{n^2} = mean\ of\ (\frac{\sigma(n)}{n} = \sum_{m|n} f(m))$.

For (a) we choose $f(n) = \frac{\mu^2(n)}{n}$ to derive $\sum \frac{\mu^2(n)}{n^2} = mean\ of\ \frac{\mu^2(n)\sigma(n)}{n}$.

However the series $\sum_p \frac{1}{p^2}$ is thus the mean of $f(n)$ given by $f(n) = \sum_{p|n} \frac{1}{p}$.

Note that $f(n)$ is not the function in Prop1 (b).

Remark

$\sum_p \frac{1}{p^2} =$
0.45224742004106549850654336483224793417323134323989…

([2], p209).This is done by summation of a suitable Dirichlet series. We obtain an upper bound (not sharp) by the following elementary comparison:

The n^{th} prime $p_n > 2n$ for $n > 4$. Hence

$$\sum_{n=5}^{\infty} \frac{1}{p_n^2} < \sum_{n=5}^{\infty} \frac{1}{(2n)^2}$$

$$= \frac{1}{4}\{\zeta(2) - \sum_{n=1}^{4} \frac{1}{p_n^2}\}$$

$$= \frac{1}{4}\{\frac{\pi^2}{6} - \frac{205}{576}\}$$

So

$$\sum_{n=1}^{\infty} \frac{1}{p_n^2} \leq \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{4}\{\frac{\pi^2}{6} - \frac{205}{576}\}$$

$$\leq 0.7437842559486$$

Remark 3We recall the estimate ([5]) for squarefree integers d:

$$\frac{L_d(1)}{L_d(2)} \leq c\prod_{p|d} (1+\frac{1}{p}) = c\frac{\sigma(d)}{d}$$

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for L -series values ; $c=\frac{5\pi^2}{48}$. In view of Proposition 1(a) we have

Corollary 1The mean of $\{\frac{L_d(1)}{L_d(2)}\}$ *is bounded above by* $\frac{25}{16}$.

Proof. By the comparison above the mean of $\{\frac{L_d(1)}{L_d(2)}\}$ is bounded above by c (mean of $\frac{\sigma(d)}{d}$). But by Prop 1(a) this bound is $c\frac{\zeta(2)}{\zeta(4)}=\frac{5\pi^2}{48}.\frac{15}{\pi^2}=\frac{25}{16}$.

Corollary 2Let $h(n)=(1-\mu^2(n))\sum_{k|n}\frac{\mu^2(k)}{k}=$ *sum of reciprocals of squarefree divisors of non squarefree* n. *The* $\{h(n)\}$ *has mean* 0.

Proof. We apply Wintner's Theorem :If $f=1*g$ and $\sum\frac{|g(n)|}{n}<\infty$ then

Mean $f=\sum\frac{g(n)}{n}$ (i.e the residue at 1 of $\zeta(s)\sum_n\frac{g(n)}{n^s}$=Mean f if $\sum\frac{|g(n)|}{n}<\infty$)

Let $g(n)=\frac{\mu^2(n)}{n}$ so that $\sum\frac{|g(n)|}{n}=\sum\frac{g(n)}{n}=\frac{\zeta(2)}{\zeta(4)}$

$$\therefore \sum\frac{f(n)}{n}=\sum_n\frac{(\sum_{k|n}\frac{\mu^2(k)}{k})}{n}$$
$$=\sum_n\frac{\mu^2(n)\frac{\sigma(n)}{n}}{n}+\sum_n\frac{h(n)}{n}$$

Now by Wintner, Mean $\{f_n\}=\frac{\zeta(2)}{\zeta(4)}$ =Mean $\{\mu^2(n)\frac{\sigma(n)}{n}\}$ by Prop 1(a) above.

Hence Mean $\{h(n)\}$= difference of means of f and $\{\mu^2(n)\frac{\sigma(n)}{n}\}$

$=\frac{\zeta(2)}{\zeta(4)}-\frac{\zeta(2)}{\zeta(4)}=0.$

Remark 4Let $a(n)=(\sum_{k|n}\frac{1}{k})(1-\mu^2(n))$ *=sum of reciprocals of divisors of non squarefree* n.

Then Mean $\{a(n)\}$ =Mean $\{\frac{\sigma(n)}{n}\}$ - Mean $\{\mu^2(n)\frac{\sigma(n)}{n}\}=\zeta(2)-\frac{\zeta(2)}{\zeta(4)}$

$=\zeta(2)(1-\frac{1}{\zeta(4)})>0$ (by Prop 1(a) above).

Yet the resticted sum $\{h(n)\}$ in Corollary 2 has mean zero.

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