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Qualitative Analysis of solutions of system of difference equations

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Abstract:
In this article the qualitative behavior of the solution form of the system of difference equations

$$y_{k+1} = \frac{-py_k z_{k-1}}{z_k - \delta} + \mu \quad z_{k+1} = \frac{qy_{k-1}z_k}{y_k - \mu} + \delta, k \in \mathrm{N}_0,$$ where the parameters p, q, δ, μ and initial values y_{-1}, z_{-1}, y_0, z_0 are non- zero real numbers is considered.

Keywords:*System of difference equations, Positive solutions.*

I. Introduction

Consider the solutions form of the system of difference equations

$$y_{k+1} = \frac{-py_k z_{k-1}}{z_k - \delta} + \mu, \qquad z_{k+1} = \frac{qy_{k-1}z_k}{y_k - \mu} + \delta, \; k \in \mathrm{N}_0 \quad , \quad (1.1)$$

where p, q, δ, μ are parameters and the initial values is $y_{-i}, z_{-i}, i = 0,1$ are non- zero real numbers. In the past few years there has been an increasing interest in the study of difference equations see [1-14] and following this trend we analyze the qualitative behavior of the solution form of the system of difference equations.

A solution which is such that $z_k - \delta \neq 0, y_k - \mu \neq 0, k \in N_0$ is called a well-defined solution of the system (1.1) the initial values and the values of p, q, δ, μ are chosen as positive numbers.

If any one of the initial values of the solutions of (1.1) is zero then the solutions are not defined.

If $k_0 \geq 1$ such that $y_{k_0} = 0,$ then (1.1), becomes

$$y_{k_0+1} = \frac{-pz_{k_0-1}y_{k_0}}{z_{k_0} - \delta} + \mu = \mu$$

Remark:1.1

Assume that $(-p_k)_{k\in \mathrm{N}_o}$ and $(q_k)_{k\in \mathrm{N}_o}$ are two sequences of real numbers. Consider the linear difference equation $z_{\frac{3k+2}{3}} = -p_k z_k + q_k, k \in N_0.$

Then, $z_k = (\prod_{i=0}^{k-1} - p_i) z_0 + \sum_{r=0}^{k-1} (\prod_{i=r+1}^{k-1} - p_i) q_r.$

$(-p_k)_{k\in \mathrm{N}_o}$ and $(q_k)_{k\in \mathrm{N}_o}$ are constants (i.e. $-p_k = -p$ and $q_k = q$ for some real numbers p and q for all $k \in \mathrm{N}_0$), then

$$z_k = \begin{cases} -(y_0 + q_k),\; p = 1, \\ -p^k y_0 + \left(\dfrac{p^k + 1}{p+1}\right) q, otherwise, k \in \mathrm{N}_0 \end{cases}$$

Assume that $\prod_{j=i}^{k} p_j = 1$ and $\sum_{j=i}^{k} p_j = o$ for all $k < i.$

2. Main Results

Theorem 2.1

Assume that $\{(y_k, z_k)\}_{k\geq -1}$ is a well- defined solution of (1.1). Then, for $k \in \mathrm{N}_0,$

$$y_{3k} = \left(\prod_{t=0}^{k-1}\left(\frac{-p}{q}\right)^{3t+1} v_k^2\right) y_0 + \sum_{r=0}^{k-1}\left(\prod_{t=0}^{k-1}\left(\frac{-p}{q}\right)^{3t+1} v_k^2\right)\left(\left(\frac{-p}{q}\right)^{r+1} \frac{y_0 - \mu}{y_{-1}} + 1\right)\mu,$$

$$y_{3k-1}=\left(\prod_{t=0}^{k-1}\left(\frac{-p}{q}\right)^{3t}v_k^2\right)y_{-1}+\sum_{r=0}^{k-1}\left(\prod_{t=r+1}^{k-1}\left(\frac{-p}{q}\right)^{3t}v_k^2\right)\left(\left(\frac{-p}{q}\right)^{r}\frac{-pz_{-1}}{z_0-\alpha}+1\right)\mu,$$

$$z_{3k}=\left(\prod_{t=0}^{k-1}\left(\frac{q}{-p}\right)^{3t+1}u_k^2\right)z_0+\sum_{r=0}^{k-1}\left(\prod_{t=r+1}^{k-1}\left(\frac{q}{-p}\right)^{3t+1}u_k^2\right)\left(\left(\frac{q}{-p}\right)^{r+1}\frac{z_0-\delta}{z_{-1}}+1\right)\mu,$$

$$z_{3k-1}=\left(\prod_{t=0}^{k-1}\left(\frac{q}{-p}\right)^{3t}u_k^2\right)z_{-1}+\sum_{r=0}^{k-1}\left(\prod_{t=r+1}^{k-1}\left(\frac{q}{-p}\right)^{3t}u_k^2\right)\left(\left(\frac{q}{-p}\right)^{r}\frac{qy_{-1}}{y_0-\mu}+1\right)\mu.$$

Proof: Rewriting (1.1) as below

$$\frac{y_{k+1}-\delta}{y_k}=\frac{-pz_{k-1}}{z_k-\delta},\ \frac{z_{k+1}-\delta}{z_k}=\frac{qy_{k-1}}{y_k-\mu}.$$

Let $v_k=\dfrac{y_k-\mu}{y_{k-1}},\ u_k=\dfrac{z_k-\delta}{z_{k-1}},\ k\in\mathrm{N}_0,$ (2.1)

Therefore, $v_{k+1}=\dfrac{-p}{u_k},u_{k+1}=\dfrac{q}{v_k},\ k\in\mathrm{N}_0,$ (2.2)

Also $v_{k+2}=\dfrac{-p}{q}v_k,\ u_{k+2}=\dfrac{q}{-p}u_k$

Therefore, for $k\in\mathrm{N}_0$,

$$v_{3k}=\left(\frac{-p}{q}\right)^{2k-1}v_k,\quad v_{3k+1}=\left(\frac{-p}{q}\right)^{2k-1}v_k,$$

$$u_{3k}=\left(\frac{q}{-p}\right)^{2k-1}u_k,\ u_{3n+1}=\left(\frac{q}{-p}\right)^{2n-1}u_n. \quad (2.3)$$

Rewriting (2.1), $y_k=v_ky_{k-1}+\mu,\ z_k=u_kz_{k-1}+\delta.$ (2.4)

From (2.3) and (2.4) we can write

$$y_{3k}=v_{3k}y_{3k-1}+\mu=\left(\frac{-p}{q}\right)^{2k-1}v_ky_{3k-1}+\mu,\ k\in\mathrm{N}_0,$$

$$y_{3k+1}=v_{3k+1}y_{3k}+\mu=\left(\frac{-p}{q}\right)^{2k-1}v_ky_{3k}+\mu,\ k\in\mathrm{N}_0,$$

$$z_{3k}=u_{3k}z_{3k-1}+\delta=\left(\frac{q}{-p}\right)^{2k-1}v_kz_{3k-1}+\delta,\ k\in\mathrm{N}_0,$$

$$z_{3k+1}=u_{3k+1}z_{3k}+\delta=\left(\frac{q}{-p}\right)^{2k-1}u_kz_{3k}+\delta,\ k\in\mathrm{N}_0$$

which gives

$$y_{3k+1}=\left(\frac{-p}{q}\right)^{4k-2}v_kv_ky_{3k-1}+\left(\frac{-p}{q}\right)^{2k-1}v_k\mu+\mu,$$

$k\in\mathrm{N}_0$, (2.5)

$$y_{3k+2}=\left(\frac{-p}{q}\right)^{4k-2}v_kv_ky_{3k}+\left(\frac{-p}{q}\right)^{2k-1}v_k\mu+\mu,$$

$k\in\mathrm{N}_0$, (2.6)

$$z_{3k+1}=\left(\frac{q}{-p}\right)^{4k-2}u_ku_kz_{3k-1}+\left(\frac{q}{-p}\right)^{2k-1}u_k\delta+\delta,$$

$k\in\mathrm{N}_0$, (2.7)

$$z_{3k+2}=\left(\frac{q}{-p}\right)^{4k-2}u_ku_kz_{3k}+\left(\frac{q}{-p}\right)^{2k-1}u_k\delta+\delta,$$

$k\in\mathrm{N}_0$ (2.8)

Put,

$A_k=y_{3k-1},\ B_k=y_{3k},\ C_k=z_{3k-1},\ D_k=z_{3k},\ k\in\mathrm{N}_0$ (2.9)

Using (2.5)-(2.8), we get

$$A_{\frac{3k+2}{3}}=\left(\frac{-p}{q}\right)^{4k-2}v_kv_kA_k+\left(\left(\frac{-p}{q}\right)^{2k-1}v_k+1\right)\mu,$$

$k\in\mathrm{N}_0$,

$$B_{\frac{3k+2}{3}}=\left(\frac{-p}{q}\right)^{4k-2}v_kv_kB_k+\left(\left(\frac{-p}{q}\right)^{2k-1}v_k+1\right)\mu,$$

$k\in\mathrm{N}_0$,

$$C_{\frac{3k+2}{3}}=\left(\frac{q}{-p}\right)^{4k-2}u_ku_kC_k+\left(\left(\frac{q}{-p}\right)^{2k-1}u_k+1\right)\delta,$$

$k\in\mathrm{N}_0$,

$$D_{\frac{3k+2}{3}}=\left(\frac{q}{-p}\right)^{4k-2}u_ku_kD_k+\left(\left(\frac{q}{-p}\right)^{2k-1}u_k+1\right)\delta,$$

$k\in\mathrm{N}_0$

From remark (1.1) and Eq.(2.9), we get

$$y_{3k-1}=\left(\prod_{t=0}^{k-1}\left(\frac{-p}{q}\right)^{4t-2}v_kv_k\right)y_{-1}+\sum_{r=0}^{k-1}\left(\prod_{t=r+1}^{k-1}\left(\frac{-p}{q}\right)^{4t-2}v_kv_k\right)\left(\left(\frac{-p}{q}\right)^{2r-1}v_k+1\right)\mu,$$

$$y_{3k}=\left(\prod_{t=0}^{k-1}\left(\frac{-p}{q}\right)^{4t-2}v_kv_k\right)y_0+\sum_{r=0}^{k-1}\left(\prod_{t=r+1}^{k-1}\left(\frac{-p}{q}\right)^{4t-2}v_kv_k\right)\left(\left(\frac{-p}{q}\right)^{2r-1}v_k+1\right)\mu,$$

$$z_{3k-1}=\left(\prod_{t=0}^{k-1}\left(\frac{q}{-p}\right)^{4t-2}u_ku_k\right)z_{-1}+\sum_{r=0}^{k-1}\left(\prod_{t=r+1}^{k-1}\left(\frac{q}{-p}\right)^{4t-2}u_ku_k\right)\left(\left(\frac{q}{-p}\right)^{2r-1}u_k+1\right)\delta,$$

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$$z_{3k} = \left(\prod_{t=0}^{k-1}\left(\frac{q}{-p}\right)^{4t-2} u_k u_k\right) z_0 + \sum_{r=0}^{k-1}\left(\prod_{t=r+1}^{k-1}\left(\frac{q}{-p}\right)^{4t-2} u_k u_k\right)\left(\left(\frac{q}{-p}\right)^{2r-1} u_k + 1\right)\delta$$

By (2.1) and (2.2), we get $v_k v_k = \frac{-pz_{k-1}(y_k - \mu)}{y_{k-1}(z_k - \delta)}$, $u_k u_k = \frac{qy_{k-1}(z_k - \delta)}{z_{k-1}(y_k - \mu)}$. Substituting these values in the above formulae we prove the theorem.

Note

Assume $\delta = \mu = 0$, $p = -1$ and $q = 1$ in (1.1) and x_{-1}, x_0 and y_0, y_{-1} are positive real numbers.

$$y_{k+1} = \frac{y_k z_{k-1}}{z_k}, z_{k+1} = \frac{y_{k-1} z_k}{y_k} \qquad , k = 0,1,2,..$$

(2.10)

Theorem 2.2

Let $\{y_k, z_k\}$ be a solution of (2.10) with initial values $x_{-1} = a, x_0 = b$ and $y_0 = c, y_{-1} = d$. Then for $k = 0,1,2,...$ the all solutions are

$$y_{4k+1} = \frac{bd}{c}, z_{4k+1} = \frac{ac}{b}$$

$$y_{4k+2} = \frac{b^2 d}{ac}, z_{4k+2} = \frac{ac^2}{bd}$$

$$y_{4k+3} = \frac{b^2 d^2}{ac^2}, z_{4k+3} = \frac{a^2 c^3}{b^2 d}$$

$$y_{4k+4} = \frac{b^3 d^2}{a^2 c^2}, z_{4k+4} = \frac{a^2 c^3}{b^2 d^2}$$

Proof:

Since the initial values of the solutions are positive the result is true for $k = 0$.Also we have the following,

$$y_{4k-3} = \frac{bd}{c}, z_{4k-3} = \frac{ac}{b}$$

$$y_{4k-2} = \frac{b^2 d}{ac}, z_{4k-2} = \frac{ac^2}{bd}$$

$$y_{4k-1} = \frac{b^2 d^2}{ac^2}, z_{4k-1} = \frac{a^2 c^3}{b^2 d}$$

$$y_{4k} = \frac{b^3 d^2}{a^2 c^2}, z_{4k} = \frac{a^2 c^3}{b^2 d^2}$$

$$y_{4k+1} = \frac{y_{4k} z_{4k-1}}{z_{4k}}; z_{4k+1} = \frac{y_{4k-1} z_{4k}}{y_{4k}}$$

$$y_{4k+2} = \frac{y_{4k+1} z_{4k}}{z_{4k+1}}; z_{4k+1} = \frac{y_{4k} z_{4k+1}}{y_{4k+1}}$$

$$y_{4k+3} = \frac{y_{4k+2} z_{4k+1}}{z_{4k+2}}; z_{4k+3} = \frac{y_{4k+1} z_{4k+2}}{y_{4k+2}}$$

$$y_{4k+4} = \frac{y_{4k+3} z_{4k+2}}{z_{4k+3}}; z_{4k+3} = \frac{y_{4k+2} z_{4k+3}}{y_{4k+3}}$$

Therefore, the proof of the theorem is complete.

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