TEST Engineering & Management

January-February 2020
ISSN: 0193-4120 Page No. 12728 - 12730

Generalizations of Singular Value Inequalities Related to Block Matrices

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Article Info
Volume 82
Page Number: 12728 - 12730
Publication Issue:
January-February 2020

Abstract:
We establish several singular value inequalities related to positive semidefinite block matrices. A generalization of arithmetic geometric mean inequality is proved.

Keywords: *Singular values, inequality, positive semidenite matrix, block matrix.*

Article History
Article Received: *18 May 2019*
Revised: *14 July 2019*
Accepted: *22 December 2019*
Publication: *24 February 2020*

1. Introduction: Let $M_n(C)$ denote the space of $n \times n$ complex matrices. For Hermitian matrices $A, B \in M_n(C)$, we write $A \geq B$ to mean $A - B \geq 0$, particularly, $A \geq 0$ indicates that A is positive semidefinite. Likewise, we write $A > 0$ to mean A is positive definite. The absolute value of a matrix $A \in M_n(C)$, denoted by $|A|$, is defined as $|A| = \sqrt{A^*A}$. The eigenvalues of $|A|$ is called the singular values of A, denoted by $s_1(A), s_2(A), \dots, s_n(A)$ and arranged as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. Note that $s_j(A) = s_j(A^*) = s_j(|A|)$, for $j = 1, \dots, n$. If A is Hermitian, we label its eigenvalues as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. Interesting relations for eigenvalues of Hermitian matrices can be obtained by Weyl's monotonicity principle, which says that if $A, B \in M_n(C)$ are Hermitian and $A \geq B$, then $\lambda_j(A) \geq \lambda_j(B)$.

The direct sum of A and B, denoted by $A \oplus B$, is defined to be the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

Bhatia and Kittaneh [3] introduced the arithmetic geometric mean inequality for singular values as follows: If $A, B \in M_n(C)$, then

$$2s_j(AB^*) \leq s_j(A^*A + B^*B), \quad j = 1, \dots, n. \qquad (1.1)$$

For positive semidefinite matrices $A, B \in M_n(C)$, Zhan [7], has proved

$$s_j(A - B) \leq s_j(A \oplus B), \quad j = 1, \dots, n. \qquad (1.2)$$

Tao [6] proved that if $A, B, C \in M_n(C)$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$2s_j(B) \leq s_j \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, \quad j = 1, \dots, n. \qquad (1.3)$$

Furthermore, Bhatia and Kittaneh [4] obtained that if $A, B \in M_n(C)$ are such that A is Hermitian, $B \geq 0$ and $\pm A \leq B$, then

$$s_j(A) \leq s_j(B \oplus B), \quad j = 1, \dots, n, \qquad (1.4)$$

and

$$s_j(AB^* + BA^*) \leq s_j\big((AA^* + BB^*) \oplus (AA^* + BB^*)\big), \quad j = 1, \dots, n, \qquad (1.5)$$

for any $A, B \in M_n(C)$.

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An equivalent inequality of (1.1) was obtained by Audeh and Kittaneh [1] says that if $A, B, C \in M_n(C)$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$s_j(B) \leq s_j(A \oplus C), \qquad j = 1, \dots, n. \qquad (1.6)$$

Recently, Buraqan and Kittaneh [5] gave a new singular value inequality for sums and direct sums of matrices as follows:

If $A, B, C, X, Y \in M_n(C)$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$s_j(X^*BY + Y^*B^*X) \leq s_j\big((X^*AX + Y^*CY) \oplus (X^*AX + Y^*CY)\big), \quad (1.7)$$

for $j = 1, \dots, n$.

In this research, we generalize inequality (1.1). Other inequalities related to products and direct sums of matrices are also given.

2. **Main results**:

To generalize inequality (1.1), we need the following lemma that relates the eigenvalues of $\begin{bmatrix} 0 & K \\ K^* & 0 \end{bmatrix}$ with the singular values of K.

Lemma 2.1 [**2**]**:** The Hermitian matrix $\begin{bmatrix} 0 & K \\ K^* & 0 \end{bmatrix}$, where $K \in M_n(C)$ with rank r, has eigenvalues

$$s_1(K), \dots, s_r(K), \underbrace{0, \dots, 0}_{2(n-r)}, -s_r(K), \dots, -s_1(K).$$

Theorem 2.1: Let $A, B, C, D \in M_n(C)$. Then

$$2s_j(A^*B \oplus C^*D) \leq s_j\big((AA^* + BB^*) \oplus (CC^* + DD^*)\big), \quad j = 1, \dots, 2n$$

Proof. Let $M = \begin{bmatrix} A & 0 & B & 0 \\ 0 & C & 0 & D \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$$N = \begin{bmatrix} 0 & 0 & A^*B & 0 \\ 0 & 0 & 0 & C^*D \\ B^*A & 0 & 0 & 0 \\ 0 & D^*C & 0 & 0 \end{bmatrix}.$$

Then

$$MM^* = \begin{bmatrix} AA^* + BB^* & 0 & 0 & 0 \\ 0 & CC^* + DD^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M^*M = \begin{bmatrix} A^*A & 0 & A^*B & 0 \\ 0 & C^*C & 0 & C^*D \\ B^*A & 0 & B^*B & 0 \\ 0 & D^*C & 0 & D^*D \end{bmatrix}$$

and

$$M^*M - 2N = \begin{bmatrix} A^*A & 0 & -A^*B & 0 \\ 0 & C^*C & 0 & -C^*D \\ -B^*A & 0 & B^*B & 0 \\ 0 & -D^*C & 0 & D^*D \end{bmatrix} = \begin{bmatrix} A & 0 & -B & 0 \\ 0 & C & 0 & -D \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} A & 0 & -B & 0 \\ 0 & C & 0 & -D \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0.$$

Now by Weyl's monotonicity principle, we have

$$2\lambda_j(N) \leq \lambda_j(M^*M), \qquad j = 1, \dots, 2n.$$

The eigenvalues of M^*M, and MM^*, are

$$s_j\big((AA^* + BB^*) \oplus (CC^* + DD^*)\big), \qquad j = 1, \dots, 2n.$$

By Lemma 2.1, the $2n$ eigenvalues of N are

$$s_j(A^*B \oplus C^*D), \qquad j = 1, \dots, 2n.$$

Therefore,

$$2s_j(A^*B \oplus C^*D) \leq s_j\big((AA^* + BB^*) \oplus (CC^* + DD^*)\big), j = 1, \dots, 2n$$

This completes the proof.

By letting $C = D = 0$ in Theorem 2.1, we have Bhatia and Kittaneh inequality (1.1).

In the following theorem, we establish another version of the Theorem 1.1.

Theorem 2.2: Let $A, B, C, D \in M_n(C)$.Then

$$s_j(A^*B \oplus C^*D) \leq s_j(|A|^2 \oplus |B|^2 \oplus |C|^2 \oplus |D|^2), \quad j = 1, \dots, 2n.$$

Proof. Let $M = \begin{bmatrix} A & 0 & B & 0 \\ 0 & C & 0 & D \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$N = \begin{bmatrix} 0 & 0 & A^*B & 0 \\ 0 & 0 & 0 & C^*D \\ B^*A & 0 & 0 & 0 \\ 0 & D^*C & 0 & 0 \end{bmatrix}.$$

Then

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$$M^*M = \begin{bmatrix} A^*A & 0 & A^*B & 0 \\ 0 & C^*C & 0 & C^*D \\ B^*A & 0 & B^*B & 0 \\ 0 & D^*C & 0 & D^*D \end{bmatrix} \geq 0. \quad (2.1)$$

And

$$M^*M - 2N = \begin{bmatrix} A^*A & 0 & -A^*B & 0 \\ 0 & C^*C & 0 & -C^*D \\ -B^*A & 0 & B^*B & 0 \\ 0 & -D^*C & 0 & D^*D \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 & -B & 0 \\ 0 & C & 0 & -D \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} A & 0 & -B & 0 \\ 0 & C & 0 & -D \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0. \quad (2.2)$$

From inequalities (2.1) and (2.2), we get

$$\begin{bmatrix} A^*A & 0 & 0 & 0 \\ 0 & C^*C & 0 & 0 \\ 0 & 0 & B^*B & 0 \\ 0 & 0 & 0 & D^*D \end{bmatrix} \geq \pm \begin{bmatrix} 0 & 0 & A^*B & 0 \\ 0 & 0 & 0 & C^*D \\ B^*A & 0 & 0 & 0 \\ 0 & D^*C & 0 & 0 \end{bmatrix}$$

By applying inequalities (1.4)), we get

$$s_j\big((A^*B \oplus C^*D) \oplus (A^*B \oplus C^*D)^*\big) \leq s_j\big((A^*A \oplus B^*B \oplus C^*C \oplus D^*D) \oplus (A^*A \oplus B^*B \oplus C^*C \oplus D^*D)\big).$$

Thus,

$$s_j\big((A^*B \oplus C^*D) \oplus (A^*B \oplus C^*D)\big) \leq s_j\big((A^*A \oplus B^*B \oplus C^*C \oplus D^*D) \oplus (A^*A \oplus B^*B \oplus C^*C \oplus D^*D)\big),$$

which is equivalent to

$$s_j(A^*B \oplus C^*D) \leq s_j(A^*A \oplus B^*B \oplus C^*C \oplus D^*D).$$

This completes the proof. ■

A singular value inequality for products and direct sums of matrices is established in the following theorem.

Theorem 2.3: Let $A, B, C, D \in M_n(C)$, Then

$$s_j(AB^* + CD^*) \leq s_j\big((AA^* + CC^*) \oplus (BB^* + DD^*)\big), \quad j = 1,2, \ldots n.$$

Proof: For any $A, B, C \in M_n(C)$, we have

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix}^* \geq 0.$$

Thus,

$$\begin{bmatrix} AA^* + CC^* & AB^* + CD^* \\ BA^* + DC^* & BB^* + DD^* \end{bmatrix} \geq 0$$

Now, using inequality (1.6), we get

$$s_j(AB^* + CD^*) \leq s_j\big((AA^* + CC^*) \oplus (BB^* + DD^*)\big), \quad j = 1,2, \ldots n.$$

This completes the proof. ■

In Theorem 2.3, by letting $C = B$ and $D = A$, we get inequality (1.5),

$$s_j(AB^* + BA^*) \leq s_j\big((AA^* + BB^*) \oplus (AA^* + BB^*)\big).$$

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