The Implementation of Bessel Function and its Importance to the Electromagnetic Waveguides

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Abstract

In this paper, Bessel functions are revisited extensively for its definitions, characteristics properties and kinds. The Bessel functions are started from single dimensional, two dimensional and multi dimensional variables in its applications. Also the various applicationsof Bessel functions are studied. The Bessel function curves for various order and arguments are extended. The diagrams were simulated and presented. These generalized Bessel function application find useful in microwave engineering of circular waveguides.

Keywords: *Bessel, differential equation, wave propagation, circular waveguide.*

I. INTRODUCTION

In circular waveguides, the Bessel functions are used. Bessel functions are formerly distincted by Daniel Bernoulli, a Swiss mathematician who worked in the chain oscillations hanged in one of the end, and Leonhard Euler, who experimented the stretched membrane vibrations. Later it was widely used by Friedrich Wilhelm Bessel, a German astronomer around 1817, for the period of doing research of solutions

For John Kepler’s equations of terrestrial motion. They are known as canonical solutions y(x) of Bessel's differential equation for an arbitrary complex number α, the order of the Bessel function. Bessel function, is also called as Cylinder Function, due to a set of mathematical functions thoroughly obtained.

After F.W. Bessel announced his results, the researchers started finding that the above functions appeared in many mathematical descriptions by the phenomena, with hea flow, electric flow in a rigid tubes, the electromagnetic waves propagation, the light diffraction, the fluids motions, and the elastic body deformations. Among these researchers, Lord Rayleigh, also kept the Bessel functions in a more expressions context by displaying that they arise in the Laplace’s equation solutions(7).But, when Laplace equation is formulated in cylindrical rather than spherical or Cartesian coordinates.

II. LITERATURE SURVEY

Many fields use Bessel functions including acoustic sound theory, electromagnetic field theory, fluid dynamics, nuclear energy physics and wireless physics. Application of this function in electromagnetic waves propagated in cylindrical waveguides, in viscid rotational flow pressure values, Circular object heat conduction, vibration modes of a thin circular sound acoustic membrane, lattice diffusion problems.[6]

Parker Kuklinski (2019) presented Bessel function properties in generalized manner. [1]

Kuklinski (2018) also published in his research work about the application of Bessel function to multi tone sine modulation waveforms [2].

Mehdi M. Molu, et al (2017) presented a new equal description of Modified Bessel Functions for doing analysis of Multi-Hop Wireless Communication Systems [3].

D.Lovri C et al (2011) analyzed in his numerical analysis of various electromagnetic problems, that computation formulas are often based on Bessel functions[4]

Frank Bowman, (2010) introduced the books on Bessel Functions. [5]

In this work, the various Bessel function order and arguments were analyzed and implemented by using simulation tool.

III. BESSEL FUNCTION

Bessel functions are defined as the radial component of cylindrical shaped drum's oscillation mode or circular waveguide. The figure 1 shows the Bessel function in circular waveguide.

IV. PROPERTIES OF BESSEL FUNCTION

The recurrence relation of general order can be solved by gamma function. In particular, mathematically, a Bessel function's property obey the solution of the following differential equations.

$$y' = dy/dx \quad (1)$$
$$y'' = d^2y/dx^2 \quad (2)$$
$$x^2y'' + xy' + (x^2 - v^2)y = 0 \quad (3)$$
$$x^2y'' + xy' + (x^2 - n^2)y = 0 \quad (4)$$

Equation 3 and 4 is denoted as Bessel's equation. The variable "v" is called as the Bessel equation's order.For integral values of the order "v" or "n", the Bessel functions are

$$J_n(x) = \frac{x^n}{2^n n!}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots(5)\right.$$

The first kind is denoted by J. The second kind is represented by Y (sometimes neuman function).The $J_0(x)$graph resembles like that of a damped cosine curve function, and that of $J_1(x)$ views as that of a damped sine curve function [8] shown in figure 2.

V. SIMULATIONS

Some specific material problems direct to differential equations similar to Bessel's equation. Hence, the solutions of them look as the appearance of combinations of Bessel functions.

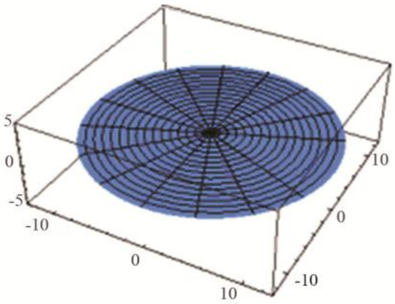

Fig.1.Bessel function in circular waveguide

They are called as second order (kind) Bessel functions or thirdorder (kind) [9].It is explained by an example.

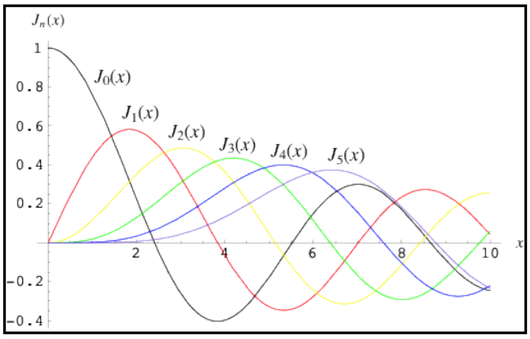

Fig.2.Bessel function curve with v=0 to 5 and x=1 to x=10

As an example, in solving the wave propagation or heat flow equations in cylindrical coordinate system, the mathematical method of separation of variables show the ways to Bessel's differential equation. This is the solution of the Bessel function, represented by $J_n(x)$.By the Calculation, first kind $J_v(x)$Bessel functions, second kind $Y_v(x)$ Bessel functions and their first derivatives $J'_v(x)$,$Y'_v(x)$ are established.

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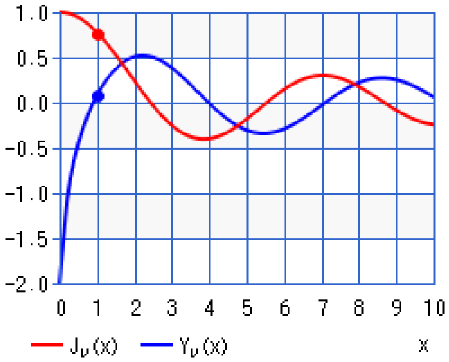

Fig.3.Bessel function curve with v=0 and x=1
$J_v x$=0.765 and $Y_v x$=0.088

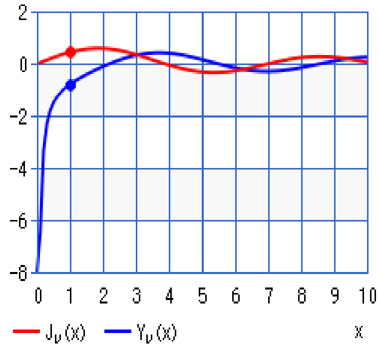

Fig.4.Bessel function curve with v=1 and x=1,
$J_v x$=0.440 $Y_v x$=-0.780

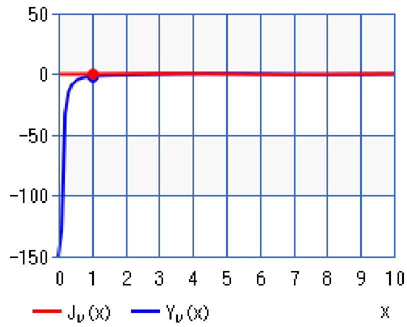

Fig.5. Bessel function curve with v=2 x=1,
$J_v x$=0.114 and $Y_v x$=-1.650

The first kind Bessel functions $J_n(x)$ or $J_v(x)$ are showed as the solutions to theBessel differential expression given in Equation4

Table 1. Bessel function in terms x and n.

x	n	J_0x	J_1x	Y_0x	Y_1x
1	0	0.765	0.440	0.088	-0.780
1	1	0.765	0.440	0.088	-0.780
1	2	0.088	0.440	0.088	-0.780

These are non-singular at the origin. These functions are occasionally also called as cylindrical functions or harmonics of cylindrical.

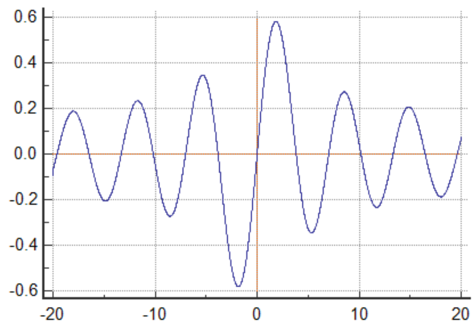

Fig.6. Bessel function curve with v=0.5 and x=2

The above plot shows $J_n(x)$ for n=0, 1, 2,..., 5. The denotion J_zn was first applied by Hansen in the year1843 and subsequently followed by Schlomilchin the year 1857 to represent as $J_n(2z)$ [10]which is given by Watson in the year 1966.

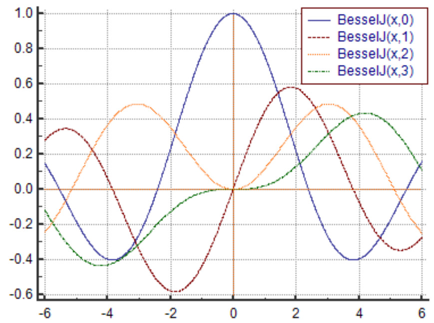

Fig.7. Bessel function of different orders and parameters of n.

But Still, Hansen's function definition it selflies in expressions of the generating function model.

The second order ordinary differential equation of type linear is known as the Bessel expression of second kind given in Equation3.

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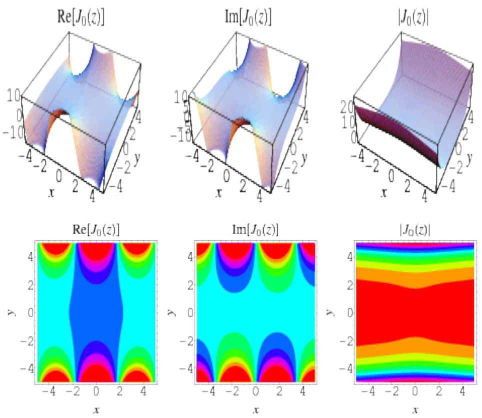

Fig.8. Simulated results of Bessel function of real and imaginary values.

The above differential equations are given by Bessel, who studied this equation in detail in the year 1824 and published its solutions. They are expressed in terms of a special class of functions called cylinder functions or Bessel functions. Actualde notion of the general solution is depending on the order v.Further it is considered as two cases: first case, the ordervis non-integer; second case, the ordervis an integer.

Case1:When the Orderis Non-Integer, i.e., v is non integer

Here it is assumed that the number is positive and non-integer. For this, the common solution of the Bessel function will be writing as

$$y(x)=C_1J_v(x)+C_2J_{-v}(x), \quad (6)$$

WhereC_{1and} $C_2$are termed as arbitrary constants and$J_v(x)$,$J_{-v}(x)$arethe first kind Bessel functions.The Bessel function may be pointed by a series of function, calledGamma function:

$$J_v(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{\Gamma(p+1)\Gamma(p+v+1)} \cdot \left(\frac{x}{2}\right)^{2p+v} \quad (7)$$

The Gamma function is the factorial function generalizationfrom integervalues to all real values. In particular, the above function has the following peculiar properties:

$$\Gamma(p+1)=p!,$$

$$\Gamma(p+v+1)=(v+1)(v+2)...(v+p)\Gamma(v+1). (8)$$

The negative order$(-v)$Bessel functions are written in similar way:

$$J_{-v}(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{\Gamma(p+1)\Gamma(p-v+1)} \cdot \left(\frac{x}{2}\right)^{2p-v} \quad (9)$$

The Bessel functions are calculated by the most mathematical software packages. For example, the Bessel functions of the1st kind of ordersv=0tov=4are shown inFigure2.

Case2:When the Order is an Integer, i.e v is integer

When the Bessel differential equation ordervis an integer, the functions$J_v(x)$and$J_{-v}(x)$can change into dependent from each other. By this case the common solution is given by some other formula:

$$y(x)=C_1J_v(x)+C_2Y_v(x) \quad (10)$$

where$Y_v(x)$is the second kind Bessel function. Often this functions areknown as Weber function, Neumann functions.

The Bessel function of the next kind can be pointed as $Y_v(x)$ through the Bessel functions of the first kind $J_v(x)$ and $J_{-v}(x)$:

$$Y_v(x)=\frac{J_v(x)\cos \pi v - J_{-v}(x)}{\sin \pi v}. \quad (11)$$

The above function graphs are displayed as$Y_v(x)$for several first ordersv in the Figure9.

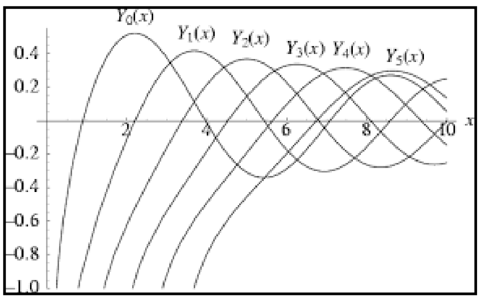

Figure 9.Bessel function second order

It is noted that in fact the differential equation general solution presented in terms of the first and second kind Bessel functions is valid for non-integer orders as well. Several Differential Equations are reducible to Bessel's Equation. Among the equations related with the Bessel's

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differential equation, the modified Bessel's equationis famous. That is obtained by replacing with−ix.Theform of that equationis given below:

$$x^2y''+xy'-(x^2+v^2)y=0 \qquad (12)$$

The solution of that expressions can be denoted as the first and second kind modified Bessel functions.

$$y(x)=C_1J_v(-ix)+C_2Y_v(-ix)=C_1I_v(x)+C_2K_v(x) \qquad (13)$$

The above$I_v(x)$is termed as modified Bessel function of first kind and $K_v(x)$ is known as modified Bessel functions of the second kind, accordingly. In astronomical science of physics, the Airy differential equation is expressed mathematically as below.

$$y''-xy=0 \qquad (14)$$

This equation could be reduced to the Bessel equation. Its solution is given as the fractional order±1/3bythe Bessel functions.

$$y(x) = C_1\sqrt{x}J_{\frac{1}{3}}\left(\frac{2}{3}ix^{\frac{3}{2}}\right) + C_2\sqrt{x}J_{\frac{-1}{3}}\left(\frac{2}{3}ix^{\frac{3}{2}}\right) \qquad (15)$$

The differential equation of type

$$x^2y''+xy'+(a^2x^2-v^2)y=0 \qquad (16)$$

varies from the Bessel equation by themultiplying factora^2in front of$x^2$and has the common solution in the form given below.

$$y(x)=C_1J_v(ax)+C_2Y_v(ax). \qquad (17)$$

The similar differential equation

$$x^2y''+axy'+(x^2-v^2)y=0 \qquad (18)$$

isshortened to the Bessel equation

$$x^2z''+xz'+(x^2-n^2)z=0 \qquad (19)$$

Where it is further simplified to by substitution method

$$y(x) = x^{\frac{1-a}{2}}z(x) \qquad (20)$$

Here the parametern^2denotes:

$$n^2=v^2+\frac{1}{4}(a-1)^2. \qquad (21)$$

As a result, the common solution for the differential equation is provided by

$$y(x) = x^{\frac{1-a}{2}}[C_1Jn(x) + C_2Yn(x)] \qquad (22)$$

The Bessel function$J_n(z)$could also be explained by the contour integral

$$J_n(z) = \frac{1}{2\pi i}\oint e^{(z/2)(t-1/t)}\, t^{-n-1}dt \qquad (23)$$

The contour integral encloses here the origin and is travel across in a anticlockwise direction given byArfkenin the year 1985.The first kind Bessel function is implemented in the Wolfram Languageas Bessel expression J[*nu*,*z*].This differential equation is solved by applying the Frobenius method. That method uses a series of the form solution.

$$y = x^k\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k} \qquad (24)$$

$$J_{-1/2\ (x)} \equiv \sqrt{\frac{2}{\pi x}}cosx \qquad (25)$$

$$J_{1/2\ (x)} \equiv \sqrt{\frac{2}{\pi x}}sinx \qquad (26)$$

Therefore, the general solution for m±1/2 is

$$y = a_0{}'J_{-1/2}\,(x) + a_1{}'J_{1/2}\,(x) \qquad (27)$$

$$J_m(x) \equiv \sum_{l=0}^{\infty}\frac{(-1)^l}{2^{2l+m}l!(m+l)!}x^{2l+m} \qquad (28)$$

Equation 28 is same as for $|m| \neq \frac{1}{2}$ and equation 25 is same for $|m| = -\frac{1}{2}$ and equation 26 is same for $|m| = \frac{1}{2}$

These $J_m(x)$and$J_{-m}(x)$ equations are related even when mis an integer as shown below.

$$J_{-m}(x) \equiv \sum_{l=0}^{\infty}\frac{(-1)^l}{2^{2l-m}l!(m-l)!}x^{2l-m} \qquad (29)$$

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From the above, it is observed that the Bessel differential equation makes as second-order, hence there must be two linearly independent solutions. It is found both only for modulus of m=±1/2. For a common nonintegral order, the independent solutions are$J_m$and J_{-m}. Whenevermis an integer, the common real solution is given in the form

$$Z_m \equiv C_1 J_m(x) + C_2 Y_m(x) \quad (30)$$

Where J_m= a first kind Bessel function,

Y_m= the second kind Bessel function,

C1andC2= constants.

Complex solutions are specified by third kind Bessel functions also called as theHankel functions.

$$e^{izcos\,\theta} = J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos(n\theta) \quad (31)$$

The Bessel function's theorem of addition states that

$$J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y) J_{n-m}(z) \quad (32)$$

Several integral functions could be mentioned in terms of Bessel functions as shown below.

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z sin\theta - n\theta)\, d\theta \quad (33)$$

This is Bessel's first integral,

$$J_n(z) = \frac{i^{-n}}{\pi}\int_0^{\pi} e^{izcos\theta} \cos(n\theta)\, d\theta \quad (34)$$

$$J_n(z) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{izcos\,\emptyset} e^{in\emptyset}\, d\emptyset \quad (35)$$

forn=1,2, ...,

$$J_n(z) = \frac{2}{\pi}\frac{z^n}{(2n-1)!}\int_0^{\pi/2} sin^{2n} u \cos(z\, cosu) du \quad (36)$$

For n=1,2, ...,

$$J_n(x) = \frac{1}{2\pi i}\int_{\gamma}^{\infty} e^{(x/2)(z-\frac{1}{z})} z^{-n-1} dz \quad (37)$$

Forthe 'n' value greater than -1/2,the Bessel functions are normalized so that

$$\int_0^{\infty} J_n(x).\, dx = 1 \quad (38)$$

for positive value of the integral and real'n'. Integrals further goes on

$J_1(x)$is including as

$$\int_0^{\infty} \left[\frac{J_1(x)}{x}\right]^2 dx = \frac{4}{3\pi} \quad (39)$$

$$\int_0^{\infty} \left[\frac{J_1(x)}{x}\right]^2 x\, dx = \frac{1}{2} \quad (40)$$

The first kind Bessel functions ratios have been continued as fraction given below.[11]

$$\frac{J_{n-1}(z)}{J_n(z)} = \frac{2n}{z} - \cfrac{\frac{z}{2(n+1)}}{1-\cfrac{\frac{\frac{z^2}{2}}{(n+1)(n+2)}}{1-\cfrac{\frac{\frac{z^2}{2}}{(n+2)(n+3)}}{1-\cdots}}} \quad (41)$$

VI. Applications

The Bessel functions are widely used in solving problems of theoretical and nuclear physics. Radio wave propagation is one such example in the investigation. Also it is utilised in heat flow conduction, oscillations of membranes in the systems with cylindrical or spherical symmetry. For Example The differential equation$x^2y''+xy'+(3x^2-2)y=0$ is solved by the following way. This equation has order$\sqrt{2}$and differs from the standard Bessel equation only by factor3beforex^2.Therefore, the general solution of the equation is expressed by the formulay $(x)=C_1J_{\sqrt{2}}(\sqrt{3}x)+C_2Y_{\sqrt{2}}(\sqrt{3}x)$,whereC1,C2are constants,$J_{\sqrt{2}}(\sqrt{3}x)$and$Y_{\sqrt{2}}(\sqrt{3}x)$are Bessel functions of thefirst andsecond kind, accordingly.

VII. Results

The Bessel functions of several order and arguments were simulated by software tools and included in the figures. The properties, characteristics and their importance is studied and tabulated. The use of Bessel function in electromagnetic wave guides in circular shape is simulated and given in figures 1-9.

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VIII. Conclusion

The Bessel functions are implemented in circular waveguides as review. The Bessel functions properties and equations are used in many other applications. Its characteristics were identified and studied. The various order values are tabulated. The importance of the research areas in circular waveguide of electro magnetics is explored.

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