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Some Fixed Point Theorems with Applications by Using a Concept of Altering Distances

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Abstract:

We attained few common fixed point theorems with application for two self maps on a complete metric space by using a concept of altering distances. These are the generalizations of the results of the Jose R.Morales and Edixon Rojas[3].

Keywords:*Complete metric space, contractive mappings, altering distances, common fixed points.*

1Introduction:

In recent times Jose R.Morales and Edixon Rojas established the following some fixed point theorems by using altering distances function.

Theorem 1.1.[3] let $\mathcal{H}$ be a complete metric space and $\mathcal{T}:\mathcal{H} \to \mathcal{H}$ be a mapping satisfying the following condition:

$$\chi\big(d(\mathcal{T}\ell,\mathcal{T}m)\big) \le \eta\big(d(\ell,m)\big)\chi\big(d(\ell,m)\big)$$

where $\chi \in$ X and $\eta: \Re_+ \to [0,1)$ with

$$\lim_{t\to u}\sup \eta(u) < 1\; \forall u > 0.$$

then $\mathscr{b}_0$ is a unique fixed point

Corollary 1.2.[3] Let $\mathcal{T}:\mathcal{H} \to \mathcal{H}$ be a mapping and$(\mathcal{H},d)$be a Complete metric space and satisfying inequality,

$$\int_0^{\chi(d(\mathcal{T}\ell,\mathcal{T}m))} \mu(u)du \le \eta\big(d(\ell,m)\big)\int_0^{\chi(d(\ell,m))} \mu(u)du$$

where $\chi \in$ X,$\mu \in$ Mand $\eta: \Re_+ \to [0,1)$ with $\lim_{t\to u}\sup \eta(u) < 1$, $\forall u > 0$.

Then$z_0 \in \mathcal{H}$ is a unique common fixed point of $\mathcal{T}$

Similarly we generalize the remaining results of Jose R.Morales and Edixon Rojas[3]

Delbosco[2] and Skof[5] presented a different idea known as altering distances

Definition 1.1.([2] and[5])

A function $\chi: \Re^+ \to \Re^+$ is said to be an altering distances function if the following properties are satisfied

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1. χ is continuous and strictly increasing on $\Re^{+}$,
2. $\chi(p) = 0$ if and only if $p = 0$ and
3. $\chi(p) \geq \mathcal{M} p^{k}$ for every $k > 0$where $\mu > 0$ and $\mathcal{M} > 0$ are constants.

For example, $\chi(p) = p^{2}$

Definition 1.2. Let $\mu: \Re_{+} \to \Re_{+}$be a function and satisfied the below conditions

1. μ is a Lebesgue integrable function on each compact subset of $\Re_{+}$.
2. μ is nonnegative
3. $\int_0^{\varepsilon} \mu(p)dp > 0$ for each $\varepsilon > 0$

Lemma 1.3.([1]and [4])

Describe $\chi_0: \Re_{+} \to \Re_{+}$ by $\chi_0(j) := \int_0^{j} \mu(p)dp$ for every $p \in \mathcal{R}_{+}$and $\mu \in$ M then $\chi_0 \in$ X.

Lemma 1.4. Consider$\mathcal{L}$be a metric space on d and $(\ell_{k})_{k}$ be a sequence in $\mathcal{L}$ such that $\lim_{k\to\infty} d(\ell_{k}, \ell_{k+1}) = 0$. If$(\ell_{k})_{k}$ is not a Cauchy sequence in $\mathcal{L}$, then $\exists$ an $\epsilon > 0$ for which we can find sub sequences $\left(\ell_{i(r)}\right)_{r}$ and $\left(\ell_{j(r)}\right)_{r}$ of $(\ell_{k})_{k}$ with $i(r) > j(r) > r$ such that $d(\ell_{i(r)}, \ell_{j(r)}) \geq \epsilon$ and $d(\ell_{i(r)-1}, \ell_{j(r)}) < \epsilon$ and

i. $\lim_{r\to\infty} d(\ell_{i(r)}, \ell_{j(r)}) = \epsilon$.

ii. $\lim_{r\to\infty} d(\ell_{i(r)-1}, \ell_{j(r)}) = \epsilon$

iii. $\lim_{r\to\infty} d(\ell_{i(r)-1}, \ell_{j(r)-1}) = \epsilon$

iv. $\lim_{r\to\infty} d(\ell_{i(r)}, \ell_{j(r)+1}) = \epsilon$

v. $\lim_{r\to\infty} d(\ell_{i(r)+1}, \ell_{j(r)+1}) = \epsilon$

vi. $\lim_{r\to\infty} d(\ell_{i(r)+1}, \ell_{j(r)+2}) = \epsilon$

2 Main results

Theorem 2.1. Let $(\mathcal{H}, d)$be a complete metric space and let $p, q: \mathcal{H} \to \mathcal{H}$ be mappings satisfying the following condition: for every $\ell, m \in \mathcal{H}$

$\chi\big(d(q\ell, pqm)\big) \leq \eta\big(d(\ell, pm)\big)\chi\big(d(\ell, pm)\big)$ (2.1)

where $\chi \in$ X and $\eta: \Re_{+} \to [0,1)$ with

$\lim_{t\to u} \sup \eta(s) < 1$, for all $u > 0$. (2.2)

the range of q contains the range of p (2.3)

and $pq = qp$ (2.4)

then p, q have a unique common fixed point.

Proof:

Since the range of q contains the range of p there exist a point $\ell_1 \in \mathcal{H} \ni q\ell_1 = p\ell_0$ here ℓ_0 be an arbitrary point.

Consider ℓ_{n+1} ,ℓ_n,ℓ_{n-1} such that $\ell_{n+1} = q\ell_n = p\ell_{n-1}$

To show$\lim_{n\to\infty} d(\ell_n, \ell_{n+1}) = 0$

Insert $\ell = \ell_{n-1}$, $m = \ell_{n-2}$ into the equation (2.1)

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$\chi\big(d(q\ell_{n-1}, pq\ell_{n-2})\big) \le \eta\big(d(\ell_{n-1}, p\ell_{n-2})\big)\chi\big(d(\ell_{n-1}, p\ell_{n-2})\big)$

$$\chi\big(d(\ell_n, \ell_{n+1})\big) \le \eta\big(d(\ell_{n-1}, \ell_n)\big)\chi\big(d(\ell_{n-1}, \ell_n)\big) < \chi\big((\ell_{n-1}, \ell_n)\big)$$

$$\chi\big(d(\ell_n, \ell_{n+1})\big) < \chi\big(d(\ell_{n-1}, \ell_n)\big) \qquad (2.5)$$

$\big(d(\ell_n, \ell_{n+1})\big)_n$ is a non-decreasing sequence.

To shown $\lim_{n\to\infty} d(\ell_n, \ell_{n+1}) = \delta \ge 0$ there exists a constant δ. Now, we are about to prove that $\delta = 0$. Assume $\delta > 0$, then Insert (2.2) in equation (2.5) and setting limits $n \to \infty$ in (2.5)

Therefore

$$0 < \chi(\delta) \le \lim_{n\to\infty} \sup \chi\big(d(\ell_n, \ell_{n+1})\big)$$
$$\le \lim_{n\to\infty} \sup\big[\eta\big(d(\ell_{n-1}, \ell_n)\big)\chi\big(d(\ell_{n-1}, \ell_n)\big)\big]$$
$$\le \lim_{n\to\infty} \sup\big[\eta\big(d(\ell_{n-1}, \ell_n)\big)\big] \lim_{n\to\infty} \sup\big[\chi\big(d(\ell_{n-1}, \ell_n)\big)\big]$$
$$\le \lim_{t\to\delta} \sup \eta(s)\chi(\delta) < \chi(\delta)$$

Which is not true for our assumption. Hence$\delta = 0$.

Implies that $\lim_{n\to\infty} d(\ell_n, \ell_{n+1}) = 0$

Now, we establish $(\ell_n)_n$ is a Cauchy sequence in $\mathcal{H}$.

Consider $(l_n)_n$ is not a Cauchy sequence, that is there exists an $\theta_0 > 0$ such that for each positive integer i, there are positive integers $a(i)$ and $b(i)$ with $a(i) > b(i) > i$ such that $d(\ell_{a(i)}, \ell_{b(i)}) \ge \theta_0$ and $d\big(\ell_{a(i)-1}, \ell_{b(i)}\big) < \theta_0$.

From Lemma 1.4 we have

$$\theta_0 = \lim_{i\to\infty} d\big(\ell_{a(i)}, \ell_{b(i)+1}\big) = \lim_{i\to\infty} d\big(\ell_{a(i)+1}, \ell_{b(i)+2}\big) \quad (2.6)$$

Insert $\ell = \ell_{a(i)}$, $m = \ell_{b(i)-1}$ in equation (2.1)

$$\chi\left(d\big(ql_{a(i)}, pql_{b(i)-1}\big)\right) \le \eta\left(d\big(l_{a(i)}, pl_{b(i)-1}\big)\right)\chi\left(d\big(l_{a(i)}, pl_{b(i)-1}\big)\right)$$
$$\chi\left(d\big(l_{a(i)+1}, l_{b(i)+2}\big)\right) \le \eta\left(d\big(l_{a(i)}, l_{b(i)+1}\big)\right)\chi\left(d\big(l_{a(i)}, l_{b(i)+1}\big)\right)$$
$$\chi\left(d\big(l_{a(i)+1}, l_{b(i)+2}\big)\right) \le \chi\left(d\big(l_{a(i)}, l_{b(i)+1}\big)\right) \qquad (2.7)$$

By inserting (2.2) and (2.6) in equation (2.7) and taking limit as $n \to \infty$ in (2.7)

$$0 \le \chi(\theta_0) = \lim_{k\to\infty} \sup \chi\left(d\big(l_{a(i)+1}, l_{b(i)+2}\big)\right)$$
$$\le \lim_{k\to\infty} sup\, \eta\left(d\big(l_{a(i)}, l_{b(i)+1}\big)\right) \lim_{k\to\infty} sup\, \chi\left(d\big(l_{a(i)}, l_{b(i)+1}\big)\right) < \chi(\theta_0)$$

This is a contradiction to our assumption,

Thus $(\ell_n)_n$ is a Cauchy sequence in the complete metric space $(\mathcal{H}, d)$. Hence there exists $y_0 \in \mathcal{H}$ such as $\lim_{n\to\infty} \ell_n = y_0$.

Now, we shown p, q have a fixed point of y_0 .

First, we prove that y_0 is a fixed point of q.

Setting $\ell = y_0$, $m = \ell_{n-2}$ in equation (2.1)

$$\chi\big(d(q\, y_0, pq\ell_{n-2})\big) \le \eta\big(d(\, y_0, p\ell_{n-2})\big)\chi\big(d(\, y_0, p\ell_{n-2})\big)$$
$$\chi\big(d(q\, y_0, \ell_{n+1})\big) \le \eta\big(d(\, y_0, \ell_n)\big)\chi\big(d(\, y_0, \ell_n)\big)$$
$$\chi\big(d(q\, y_0, \ell_{n+1})\big) \le \chi\big(d(\, y_0, \ell_n)\big)$$

where $\chi\big(d(\, y_0, \ell_n)\big) \to 0$ as $n \to \infty$ i.e., $\lim_{n\to\infty} \chi\big(d(q\, y_0, \ell_{n+1})\big) = 0$.

Since $\chi \in$ X we have that $\lim_{n\to\infty} d(q\, y_0, \ell_{n+1}) = 0$.

$$\Rightarrow d(q\, y_0, y_0) = 0$$

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$$q\, y_0 = y_0$$

That is y_0 is a fixed point of q.
To show y_0 is a fixed point of p.
Insert $\ell = \ell_n$, $m = y_0$ in equation (2.1)
$\chi\big(d(q\ell_n, pqy_0)\big) \le \eta\big(d(\ell_n, py_0)\big)\chi\big(d(\ell_n, py_0)\big)$
$\chi\Big(d\big(\ell_{n+1,} pqy_0\big)\Big) \le \eta\big(d(\ell_n, py_0)\big)\chi\big(d(\ell_n, py_0)\big)$
$\chi\Big(d\big((\ell_{n+1}, py_0)\big)\Big) \le \chi\big(d(\ell_n, y_0)\big)$
where $\chi\big(d(\ell_n, y_0)\big) \to 0$ as $n \to \infty$ i.e., $\lim_{n\to\infty} \chi\big(d(p\, y_0, \ell_{n+1})\big) = 0$.
Since $\chi \in$ X we have that $\lim_{n\to\infty} d(py_0, y_0) = 0$.

$$\Rightarrow d(py_0, y_0) = 0$$
$$py_0 = y_0$$

Therefore y_0 is a fixed point of p.
y_0 is a common fixed point of p and q.
Now we show y_0 is a unique common fixed point of p and q.
Suppose $z_0 \in \mathcal{H}$ is another common fixed point of p and q.
Insert $\ell = y_0$, $m = z_0$ into the equation (2.1)

$$\chi\big(d(qy_0, pqz_0)\big) \le \eta\big(d(y_0, pz_0)\big)\chi\big(d(y_0, pz_0)\big)$$

$0 \le \chi\big(d(qy_0, pqz_0)\big) \le \eta\big(d(y_0, pz_0)\big)\chi\big(d(y_0, pz_0)\big)$
Case 1: If $\chi\big(d(qy_0, pqz_0)\big) = 0$ then $\chi\big(d(y_0, z_0)\big) = 0$
Since $\chi \in$ X we have that $d(y_0, z_0) = 0 \Rightarrow y_0 = z_0$
Case 2:If$\chi\big(d(qy_0, pqz_0)\big) > 0$
$0 \le \chi\big(d(qy_0, pqz_0)\big) \le \eta\big(d(y_0, pz_0)\big)\chi\big(d(y_0, pz_0)\big)$

$$\chi\big(d(y_0, z_0)\big) \le \eta\big(d(y_0, z_0)\big)\chi\big(d(y_0, z_0)\big)$$
$$\chi\big(d(y_0, z_0)\big) < \chi\big(d(y_0, z_0)\big)$$

Therefore this is a contradiction to our assumption

$$\chi\big(d(qy_0, pqz_0)\big) = 0 \Rightarrow \chi\big(d(y_0, z_0)\big) = 0$$

Since $\chi \in$ X we have that $d(y_0, z_0) = 0 \Rightarrow y_0 = z_0$
From cases 1& 2
y_0 is a Unique common fixed point of p and q.

Application 1: The following is an application to Theorem 2.1

Corollary 2.2. Let $\mathcal{H}$be a complete metric space on d and let $p, q: \mathcal{H} \to \mathcal{H}$ be mappings satisfying the following condition:

$$\int_0^{\chi(d(q\ell, pqm))} \mu(u)du \ \le \eta\big(d(\ell, pm)\big) \int_0^{\chi(d(\ell, pm))} \mu(u)du (2.8)$$

where $\chi \in$ X,$\mu \in$ Mand $\eta: \Re_+ \to [0,1)$ with
$\lim_{t\to u} \sup \eta(s) < 1, \forall u > 0$.
then p, q have a unique common fixed point $y_0 \in \mathcal{H}$.

Proof. We define $\chi_0: \Re_+ \to \Re_+$ by $\chi_0(h) = \int_0^h \mu(u)du$ for $\in$ M , then $\chi_0 \in$ X . We can write (2.8) in the form

$\chi_0\Big(\chi\big(d(q\ell, pqm)\big)\Big) \le \eta\big(d(\ell, pm)\big)\chi_0\Big(\chi\big(d(\ell, pm)\big)\Big)$
$\chi_1\big(d(q\ell, pqm)\big) \le \eta\big(d(\ell, pm)\big)\chi_1\big(d(\ell, pm)\big)$

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Where $\chi_1 = \chi_0 0\chi \in \chi$. Therefore, we obtain $y_0 \in \mathcal{H}$ is a unique common fixed point of $\mathcal{p}, \mathcal{q}$by theorem 2.1.

Theorem 2.3. Let $p, q: H \to H$ be mappings and H be a complete metric space on d and satisfying the following condition:

$\chi\big(d(ql, pqm)\big) \leq \eta\big(d(l, pm)\big)\chi\big(d(l, ql)\big) + \zeta\big(d(l, pm)\big)\chi\big(d(pm, pqm)\big)$(2.9)

where $\chi \in$ X and $\eta, \zeta: \Re_+ \to [0,1)$ with

$$\left.\begin{array}{c}\eta(t) + \zeta(t) < 1 \text{for all } u \in \Re_+, \lim_{t\to 0^+} \sup \zeta(t) < 1 \\ \lim_{t\to u^+} \sup \frac{\eta(s)}{1-\zeta(s)} < 1 \text{, for all } u > 0\end{array}\right\} (2.10)$$

then $p,\ q$ have a unique common fixed point $y_0 \in H$ such that for each $y \in H$

Proof. Consider the sequence $(h_n)_n$ defined

$h_{n+1} = qh_n = fh_{n-1}$,$n = 1,2, \dots.$. where $h \in H$ be an arbitrary point

To show$\lim_{n\to\infty} d(h_n, h_{n+1}) = 0$

Put $l = h_{n-1}$, $m = h_{n-2}$ in equation (2.9)

$$\chi\big(d(qh_{n-1}, pqh_{n-2})\big) \leq \eta\big(d(h_{n-1}, ph_{n-2})\big)\chi\big(d(h_{n-1}, qh_{n-1})\big)$$
$$+\zeta\big(d(h_{n-1}, ph_{n-2})\big)\chi\big(d(ph_{n-2}, pqh_{n-2})\big)$$
$$\chi\big(d(qh_{n-1}, pqh_{n-2})\big) \leq \eta\big(d(h_{n-1}, h_n)\big)\chi\big(d(h_{n-1}, h_n)\big)$$
$$+\zeta\big(d(h_{n-1}, h_n)\big)\chi\big(d(h_n, h_{n+1})\big)$$
$$\chi\big(d(h_n, h_{n+1})\big) - \zeta\big(d(h_{n-1}, h_n)\big)\chi\big(d(h_n, h_{n+1})\big) \leq \eta\big(d(h_{n-1}, h_n)\big)\chi\big(d(h_{n-1}, h_n)\big)$$
$$\chi\big(d(h_n, h_{n+1})\big)\big[1 - \zeta\big(d(h_{n-1}, h_n)\big)\big] \leq \eta\big(d(h_{n-1}, h_n)\big)\chi\big(d(h_{n-1}, h_n)\big)$$
$$\chi\big(d(h_n, h_{n+1})\big) \leq \frac{\eta\big(d(h_{n-1}, h_n)\big)}{\big[1 - \zeta\big(d(h_{n-1}, h_n)\big)\big]}\chi\big(d(h_{n-1}, h_n)\big)$$

Now, from (2.10) we obtain

$$\chi\big(d(h_n, h_{n+1})\big) < \chi\big(d(h_{n-1}, h_n)\big)\forall n \in z_+$$

from Theorem 2.1, we determine that the $\big(d(h_n, h_{n+1})\big)_n$ is non-increasing and converges to 0.

i.e., $\lim_{n\to\infty} d(h_n, h_{n+1}) = 0$ (2.11)

Now, we Prove that $(h_n)_n$ is a Cauchy sequence in H. Assume that $(h_n)_n$ is not a Cauchy sequence, that is $d(h_{a(i)}, h_{b(i)}) \geq \theta_0$ and $d\big(h_{a(i)-1}, h_{b(i)}\big) < \theta_0$ there are positive integers $a(i)$ and $b(i)$ with $a(i) > b(i) > i$

From Lemma 1.4 we have

$\theta_0 = \lim_{i\to\infty} d\big(h_{a(i)+1}, h_{b(i)+2}\big)$ and from (2.9) we get that

$$\chi(\theta_0) = \lim_{i\to\infty} sup\, \chi\left(d\big(h_{a(i)+1}, h_{b(i)+2}\big)\right)$$
$$\chi(\theta_0) = \lim_{i\to\infty} sup\, \chi\left(d\big(qh_{a(i)}, h_{b(i)-1}\big)\right)$$
$$\leq \lim_{i\to\infty} Sup\left(\eta\left(d\big(h_{a(i)}, h_{b(i)+1}\big)\right)\chi\left(d\big(h_{a(i)}, h_{b(i)+1}\big)\right) + \zeta\left(d\big(h_{a(i)}, h_{b(i)+1}\big)\right)\chi\left(d\big(h_{a(i)+1}, h_{b(i)+2}\big)\right)\right)$$
$$\leq \lim_{i\to\infty} Sup\left(\eta\left(d\big(h_{a(i)}, h_{b(i)+1}\big)\right)\right)\lim_{i\to\infty} Sup\left(\chi\left(d\big(h_{a(i)}, h_{a(i)+1}\big)\right)\right)$$
$$+ \lim_{i\to\infty} Sup\left(\zeta\left(d\big(h_{a(i)}, h_{b(i)+1}\big)\right)\right)\lim_{i\to\infty} Sup\left(\chi\left(d\big(h_{b(i)+1}, h_{b(i)+2}\big)\right)\right)$$
$$\chi(\theta_0) = 0$$

This is not true

Hence $(h_n)_n$ is a Cauchy sequence in the complete metric space (H, d).

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Now, we prove that y_0 is a fixed point of p and q.
First, we show p has a fixed point of y_0 .
Insert $l = y_0$, $m = h_{n-2}$ in equation (2.9)

$$\chi\big(d(y_0, pqh_{n-2})\big) \le \eta\big(d(y_0, ph_{n-2})\big)\chi\big(d(y_0, qy_0)\big) + \zeta\big(d(y_0, ph_{n-2})\big)\chi\big(d(ph_{n-2}, pqh_{n-2})\big)$$

$$\chi\big(d(q\,y_0, h_{n+1})\big) \le \eta\big(d(y_0, h_n)\big)\chi\big(d(y_0, q\,y_0)\big) + \zeta\big(d(y_0, h_n)\big)\chi\big(d(h_n, h_{n+1})\big)$$

$$\lim_{n\to\infty} Sup\,\chi\big(d(q\,y_0, h_{n+1})\big) \le \lim_{n\to\infty} Sup\,\eta\big(d(y_0, h_n)\big)\chi\big(d(y_0, q\,y_0)\big) + \lim_{n\to\infty} Sup\,\zeta\big(d(y_0, h_n)\big)\chi\big(d(h_n, h_{n+1})\big)$$

$$0 < \chi\big(d(q\,y_0,\ y_0)\big) < \chi\big(d(q\,y_0,\ y_0)\big)$$

Which is a contradiction. Thus $d(q\,y_0,\ y_0) = 0$

$q\,y_0 = y_0$.

Now we show y_0 is a fixed point of p.
Setting $l = h_n$, $m = y_0$ in equation (2.9)

$$\chi\big(d(q\,h_n, pqy_0)\big) \le \eta\big(d(h_n, py_0)\big)\chi\big(d(h_n, qh_n)\big) + \zeta\big(d(h_n, py_0)\big)\chi\big(d(py_0, pqy_0)\big)$$

$$\chi\big(d(h_{n+1}, py_0)\big) \le \eta\big(d(h_n, py_0)\big)\chi\big(d(h_n, h_{n+1})\big) + \zeta\big(d(h_n, pz_0)\big)\chi\big(d(py_0, py_0)\big)$$

$$\lim_{n\to\infty} Sup\,\chi\big(d(h_{n+1}, py_0)\big) \le \lim_{n\to\infty} Sup\,\eta\big(d(h_n, py_0)\big)\chi\big(d(h_n, h_{n+1})\big) + \lim_{n\to\infty} Sup\zeta\big(d(h_n, py_0)\big)\chi\big(d(py_0, py_0)\big)$$

$$0 < \chi\big(d(p\,y_0,\ y_0)\big) \le 0$$

Thus $d(p\,y_0,\ y_0) = 0$

$p\,y_0 = y_0$.

y_0 is a fixed point p

∴ y_0 is a common fixed point p and q.

Now prove that y_0 is a unique common fixed point p and q.
Suppose $z_0 \in H$ is another common fixed point of p and q.
Put $l = y_0$, $m = z_0$ in equation (2.9)

$$\chi\big(d(q\,y_0, pqz_0)\big) \le \eta\big(d(y_0, pz_0)\big)\chi\big(d(y_0, qy_0)\big) + \zeta\big(d(y_0, pz_0)\big)\chi\big(d(pz_0, pqz_0)\big)0$$
$$< \chi\big(d(q\,y_0, pqz_0)\big) \le \eta\big(d(y_0, pz_0)\big)\chi\big(d(y_0, y_0)\big) + \zeta\big(d(y_0, pz_0)\big)\chi\big(d(z_0, z_0)\big)$$
$$0 < \chi\big(d(q\,y_0, pqz_0)\big) \le 0$$
$$d(y_0, z_0) = 0 \Longrightarrow y_0 = z_0$$

Therefore y_0 is a unique common fixed point of p& q

Application 2: The following is an application to the Theorem 2.3

Corollary 2.4. Let (H, d)be a Complete metric space and let $p, q: H \to H$ be mappings satisfying the following condition:

$$\int_0^{\chi(d(pl,pqm))} \mu(u)du \le \eta\big(d(l,pm)\big)\int_0^{\chi(d(l,ql))} \mu(u)du + \zeta\big(d(l,pm)\big)\int_0^{\chi(d(pm,pqm))} \mu(u)du \tag{2.12}$$

where $\chi \in$ X,$\mu \in$ Mand $\eta, \zeta: \Re_+ \to [0,1)$ with

$$\left.\begin{array}{c}\eta(u) + \zeta(u) < 1 \text{for all } u \in \Re_+, \lim\limits_{t\to 0^+} \sup \zeta(t) < 1 \\ \lim\limits_{t\to u^+} \sup \dfrac{\eta(s)}{1-\zeta(s)} < 1 \text{, for all } u > 0\end{array}\right\}$$

then $p,\ q$ have a unique common fixed point $y_0 \in H$.

Proof. We define $\chi_0: \Re_+ \to \Re_+$ by $\chi_0(h) = \int_0^h \mu(u)du$ for $\in$ M , then $\chi_0 \in$ X and so inequality (2.12) becomes

$$\chi_0\left(\chi\big(d(ql,pqm)\big)\right) \le \eta\big(d(l,pm)\big)\chi_0\left(\chi\big(d(l,ql)\big)\right) + \zeta\big(d(l,pm)\big)\chi_0\left(\chi\big(d(pm,pqm)\big)\right)$$

$$\chi_1\big(d(ql,pqm)\big) \le \eta\big(d(l,pm)\big)\chi_1\big(d(l,pm)\big)$$

Where $\chi_1 = \chi_0 0\chi \in \chi$. Hence, from Theorem 2.3 we conclude that p, q have a unique common fixed point $y_0 \in H$.

Theorem 2.5. Let (H,d) be a complete metric space and let $p,q: H \to H$ be mappings satisfying the following condition: for every $l,m \in H$

$$\chi\big(d(ql,pqm)\big) \le \eta\big(d(l,pm)\big)\big[\chi\big(d(l,ql)\big) + \chi\big(d(pm,pqm)\big)\big] \tag{2.13}$$

where $\chi \in$ X and $\zeta: \Re_+ \to [0,\frac{1}{2})$ with

$$\lim_{t\to u^+} \sup \frac{\zeta(s)}{1-\zeta(s)} < 1, \text{for all } u > 0$$

then $p,\ q$ have a unique common fixed point $y_0 \in H$.

Proof. Let $h \in H$ be an arbitrary point, and Let the sequence $(h_n)_n$ defined

$h_{n+1} = qh_n = ph_{n-1}$,$n = 1,2, \dots ..$

To show$\lim_{n\to\infty} d(h_n, h_{n+1}) = 0$

Insert $l = h_{n-1}$, $m = h_{n-2}$ in equation (2.13)

$$\chi\big(d(qh_{n-1}, pqh_{n-2})\big) \le \zeta\big(d(h_{n-1}, ph_{n-2})\big)\big[\chi\big(d(h_{n-1}, qh_{n-1})\big) + \chi\big(d(ph_{n-2}, pqh_{n-2})\big)\big]$$

$$\chi\big(d(h_n, h_{n+1})\big) \le \zeta\big(d(h_{n-1}, h_n)\big)\big[\chi\big(d(h_{n-1}, h_n)\big) + \chi\big(d(h_n, h_{n+1})\big)\big]$$

$$\chi\big(d(h_n, h_{n+1})\big) - \zeta\big(d(h_{n-1}, h_n)\big)\chi\big(d(h_n, h_{n+1})\big) \le \zeta\big(d(h_{n-1}, h_n)\big)\chi\big(d(h_{n-1}, h_n)\big)$$

$$\chi\big(d(h_n, h_{n+1})\big)\big[1 - \zeta\big(d(h_{n-1}, h_n)\big)\big] \le \zeta\big(d(h_{n-1}, h_n)\big)\chi\big(d(h_{n-1}, h_n)\big)$$

$$\chi\big(d(h_n, h_{n+1})\big) \le \frac{\zeta\big(d(h_{n-1}, h_n)\big)}{\big[1 - \zeta\big(d(h_{n-1}, h_n)\big)\big]}\chi\big(d(h_{n-1}, h_n)\big)$$

$$\chi\big(d(h_n, h_{n+1})\big) < \chi\big(d(h_{n-1}, h_n)\big)\forall n \in z_+$$

as in the proof of Theorem 2.1, we conclude that the $\big(d(h_n, h_{n+1})\big)_n$ is non-increasing and converges to 0.

i.e., $\lim_{n\to\infty} d(h_n, h_{n+1}) = 0 \quad (2.14)$

Consider $(h_n)_n$ is not a Cauchy sequence, That means there are positive integers $a(j)$ and $b(j)$ with $a(j) > b(j) > j$ such that $d(h_{a(j)}, h_{b(j)}) \ge \theta_0$ and $d\big(h_{a(j)-1}, h_{b(j)}\big) < \theta_0$.

From Lemma 1.4 we have

$\theta_0 = \lim_{k\to\infty} d\big(h_{a(j)+1}, h_{b(j)+2}\big)$ and from (2.13) we get that

$$\chi(\theta_0) = \lim_{k\to\infty} sup\chi\left(d\big(h_{a(j)+1}, h_{b(j)+2}\big)\right)$$

$$\le \lim_{k\to\infty} Sup\left(\zeta\left(d\big(h_{a(j)+1}, h_{b(j)+1}\big)\right)\left[\chi\left(d\big(h_{a(j)}, h_{a(j)+1}\big)\right) + \chi\left(d\big(h_{b(j)+1}, h_{b(j)+2}\big)\right)\right]\right)$$

$$\le \lim_{k\to\infty} Sup\left(\zeta\left(d\big(h_{a(j)+1}, h_{b(j)+1}\big)\right)\chi\left(d\big(h_{a(j)}, h_{a(j)+1}\big)\right) + \zeta\left(d\big(h_{a(j)}, h_{b(j)+1}\big)\right)\chi\left(d\big(h_{b(j)+1}, h_{b(j)+2}\big)\right)\right)$$

$$= \lim_{k\to\infty} Sup\left(\zeta\left(d\big(h_{a(j)+1}, h_{b(j)+1}\big)\right)\right)\lim_{k\to\infty} Sup\left(\chi\left(d\big(h_{a(j)+1}, h_{a(j)+1}\big)\right)\right)$$

$$+ \lim_{k\to\infty} Sup\left(\zeta\left(d\big(h_{a(j)}, h_{b(j)+1}\big)\right)\right)\lim_{k\to\infty} Sup\left(\zeta\left(d\big(h_{b(j)+1}, h_{b(j)+2}\big)\right)\right)$$

$$\chi(\theta_0) = 0$$

This is a contrary to our assumption

Thus $(h_n)_n$ is a Cauchy sequence in the complete metric space (H, d). Hence there exists $y_0 \in H$ such that $\lim_{n\to\infty} h_n = y_0$.

Now, we prove that y_0 is a fixed point of p and q.

First, we show y_0 is a fixed point of q.

Setting $l = y_0$, $m = h_{n-2}$ in equation (2.13)

$$\chi(d(q\, y_0, pqh_{n-2})) \le \zeta(d(y_0, ph_{n-2}))[\chi(v(y_0, qy_0)) + \chi(d(ph_{n-2}, pqh_{n-2}))]$$

$$\chi(d(q\, y_0, h_{n+1})) \le \zeta(d(y_0, h_n))\chi(d(y_0, q\, y_0)) + \zeta(d(y_0, h_n))\chi(d(h_n, h_{n+1}))$$

$$\lim_{n\to\infty} Sup\, \chi(d(q\, y_0, h_{n+1})) \le \lim_{n\to\infty} Sup\, \zeta(d(y_0, h_n))\chi(d(y_0, q\, y_0)) + \lim_{n\to\infty} Sup\, \zeta(d(y_0, h_n))\chi(d(h_n, h_{n+1}))$$

$$0 < \chi(d(q\, y_0,\, y_0)) < \chi(d(q\, y_0,\, y_0))$$

Which is a contradiction. Thus $d(q\, y_0,\, y_0) = 0$

$q\, y_0 = y_0$.

Now we show y_0 is a fixed point of p.

Setting $l = h_n$, $m = y_0$ in equation (2.13)

$$\chi(d(q\, h_n, pqy_0)) \le \zeta(d(h_n, py_0))[\chi(d(h_n, qh_n)) + \chi(d(py_0, pqy_0))]$$

$$\chi(d(h_{n+1}, py_0)) \le \zeta(d(h_n, py_0))\chi(d(h_n, h_{n+1})) + \zeta(d(h_n, py_0))\chi(d(py_0, py_0))$$

$$\lim_{n\to\infty} Sup\, \chi(d(h_{n+1}, py_0)) \le \lim_{n\to\infty} Sup\, \zeta(d(h_n, py_0))\chi(d(h_n, h_{n+1})) + \lim_{n\to\infty} Sup \zeta(d(h_n, py_0))\chi(d(py_0, py_0))$$

$$0 < \chi(d(p\, y_0,\, y_0)) \le 0$$

Which is a contradiction. Thus $d(p\, y_0,\, y_0) = 0$

$p\, y_0 = y_0$.

y_0 is a fixed point p

$\therefore$ y_0 is a common fixed point p and q.

and Now prove that y_0 is a unique common fixed point p and q.

Suppose $z_0 \in H$ is another common fixed point of p and q.

Setting $l = y_0$, $m = z_0$ in equation (2.13)

$$\chi(d(q\, y_0, pqz_0)) \le \zeta(d(y_0, pz_0))[\chi(d(y_0, qy_0)) + \chi(d(pz_0, pqz_0))]$$

$$0 < \chi(d(q\, y_0, pqz_0)) \le \zeta(d(y_0, pz_0))\chi(d(y_0, y_0)) + \zeta(d(y_0, pz_0))\chi(d(z_0, z_0))$$

$$0 < \chi(d(q\, y_0, pqz_0)) \le 0$$

$d(y_0, z_0) = 0 \Longrightarrow y_0 = z_0$

Therefore y_0 is a unique common fixed point of & q .

Application 3: The following is an application to the Theorem 2.5

Corollary 2.6.Let (H, d)be a complete metric space and let $p, q\colon H \to H$ be mapping satisfying the following condition:

$$\int_0^{\chi(d(ql,pqm))} \mu(u)du \le \zeta(d(l, pm)) \left[\int_0^{\chi(d(l,ql))} \mu(u)du + \int_0^{\chi(d(pm,pqm))} \mu(u)du\right] \tag{2.15}$$

where $\chi \in$ X and $\zeta\colon \Re_+ \to [0, \frac{1}{2})$ with

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$$\lim_{t\to u^+} \sup \frac{\zeta(s)}{1-\zeta(s)} < 1\text{, for all } u > 0$$

then $p,\ q$ have a unique common fixed point $y_0 \in H$.

Proof. We define $\chi_0: \Re_+ \to \Re_+$ by $\chi_0(h) = \int_0^h \mu(u)du$ for $\in$ M , then $\chi_0 \in$ X and so inequality (2.15)becomes

$$\chi_0\left(\chi\big(d(ql, pqm)\big)\right) \leq \zeta\big(d(l, m)\big)\left[\chi_0\left(\chi\big(d(l, ql)\big)\right) + \chi_0\left(\chi\big(d(pm, pqm)\big)\right)\right]$$

where $\chi_1 = \chi_0 0\chi \in \chi$. Hence, from Theorem 2.5 we conclude that $p,\ q$ have a unique fixed point $y_0 \in H$.

Theorem 2.7.: Let (H, d) be a complete metric space and let $p, q\ :\ H \to H$ be continuous mappings. We denote $w(l, pm) = max\left\{\frac{d(l,ql)d(pm,qpm)}{d(l,pm)}, d(l, pm)\right\} \forall\ l, m \in H\ , l \neq m.$ (2.16)

Suppose that p, q satisfies the following condition:

$$\chi\big(d(ql, pqm)\big) \leq \eta\big(d(l, pm)\big)\chi\big(w(l, pm)\big) \qquad (2.17)$$

$\forall\ l, m \in H\ , \chi \in$ X and $\eta: \Re \to [0,1)$is a function with $\lim_{t\to u} sup\eta(t) < 1\ \forall\ u > 0$ (2.18)

then p, q have a unique common fixed point $y_0 \in H$.

Proof: Let $h_0 \in H$ be an arbitrary point and we define the sequence $h_{n+1} = qh_n = ph_{n-1}$,

$$n = 1,2, \dots \dots ....$$

It follows from (2.17) that

$$\chi\big(d(h_n, h_{n+1})\big) = \chi\big(d(qh_{n-1}, pqh_{n-2})\big)$$

$$\leq \eta\big(d(h_{n-1}, ph_{n-2})\big)\chi\big(w(h_{n-1}, ph_{n-2})\big)$$

$$\leq \eta\big(d(h_{n-1}, h_n)\big)\chi\big(w(h_{n-1}, h_n)\big)$$

By using (2.16) we get

$$w(h_{n-1}, h_n) = w(h_{n-1}, ph_{n-2})$$

$$= max\left\{\frac{d(h_{n-1}, qh_{n-1})d(ph_{n-2}, qph_{n-2})}{d(h_{n-1}, ph_{n-2})}, d(h_{n-1}, ph_{n-2})\right\}$$

$$= max\left\{\frac{d(h_{n-1}, h_n)d(h_n, h_{n+1})}{d(h_{n-1}, h_n)}, d(h_{n-1}, h_n)\right\}$$

$$= max\{d(h_n, h_{n+1}), d(h_{n-1}, h_n)\},$$

which implies,

$$\chi\big(d(h_n, h_{n+1})\big) \leq \eta\big(d(h_{n-1}, h_n)\big)\chi(max\{d(h_n, h_{n+1}), d(h_{n-1}, h_n)\})$$

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from (2.18) we have

$\chi\big(d(h_n, h_{n+1})\big) < \chi(max\{d(h_n, h_{n+1}), d(h_{n-1}, h_n)\})$

Case 1: If $d(h_n, h_{n+1}) > d(h_{n-1}, h_n)$ then $\chi\big(d(h_n, h_{n+1})\big) < \chi\big(d(h_n, h_{n+1})\big)$

Which is not true

Therefore $max\{d(h_n, h_{n+1}), d(h_{n-1}, h_n)\}= d(h_{n-1}, h_n)$

$\chi\big(d(h_n, h_{n+1})\big) < \chi(d(h_{n-1}, h_n))$

Then, it follows that $\big(d(h_n, h_{n+1})\big)_n$ is a monotone decreasing sequence of numbers consequently, there exists $\delta \geq 0$ such that $\lim_{n\to\infty} d(h_n, h_{n+1}) = \delta$. Suppose that $\delta > 0$, then

$$0 < \chi(\delta) \leq \chi\big(d(h_n, h_{n+1})\big) < \chi(d(h_{n-1}, h_n))$$

Taking limits as $n \to \infty$ above inequality yields $\chi(\delta) < \chi(\delta)$ which is a contradiction. Therefore $\delta = 0$, thus $\lim_{n\to\infty} d(h_n, h_{n+1}) = 0$ (2.19)

Now, we show that $(h_n)_n$ is a Cauchy sequence in H. Assume that $(h_n)_n$ is not a Cauchy sequence, then there exists a θ_0 and subsequences $\big(h_{a(\not{f})}\big)_{\not{f}}, \big(h_{b(\not{f})}\big)_{\not{f}}$ of $(h_n)_n$ with $a(\not{f}) > b(\not{f}) > f$ such that $d\big(h_{a(\not{f})}, h_{b(\not{f})}\big) \geq \theta_0$ and $d\big(h_{a(\not{f})-1}, h_{b(\not{f})}\big) < \theta_0$.

from Lemma 1.3 we have

$\lim_{k\to\infty} d\big(h_{a(\not{f})-1}, h_{b(\not{f})-1}\big) = \theta_0$ (2.20)

Inequality (2.17) gives us

$$\chi(\theta_0) \leq \chi\left(d\big(h_{a(\not{f})}, h_{b(\not{f})}\big)\right) = \chi\big(d(qh_{a(\not{f})-1}, pqh_{b(\not{f})-3})\big)$$

$$\leq \eta\big(d(h_{a(\not{f})-1}, ph_{b(\not{f})-3})\big)\chi\big(w(h_{a(\not{f})-1}, ph_{b(\not{f})-3})\big)$$

$$\leq \eta\big(d(h_{a(\not{f})-1}, h_{b(\not{f})-1})\big)\chi\big(w(h_{a(\not{f})-1}, h_{b(\not{f})-1})\big)$$

On the other hand, we have

$$w\big(h_{a(\not{f})-1}, h_{b(\not{f})-1}\big) = w(h_{a(\not{f})-1}, ph_{b(\not{f})-3})$$

$$= max\left\{\frac{d\big(h_{a(\not{f})-1}, qh_{a(\not{f})-1}\big)d(ph_{b(\not{f})-3}, qph_{b(\not{f})-3})}{d(h_{a(\not{f})-1}, ph_{b(\not{f})-3})}, d(h_{a(\not{f})-1}, ph_{b(\not{f})-3})\right\}$$

$$= max\left\{\frac{d\big(h_{a(\not{f})-1}, h_{a(\not{f})}\big)d(ph_{b(\not{f})-1}, h_{b(\not{f})})}{d(h_{a(\not{f})-1}, h_{b(\not{f})-1})}, d(h_{a(\not{f})-1}, h_{b(\not{f})-1})\right\}$$

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Now, by taking Upper limit as $k \to \infty$ and using (2.19) and (2.20), we have

$$\chi(\theta_0) < \chi(max(0, \theta_0)) = \chi(\theta_0)$$

Which is a contradiction. Hence $(h_n)_n$ is a Cauchy sequence in the complete metric space (H, d).

Thus, there exist $y_0 \in H$ such that $\lim_{n\to\infty} h_n = y_0$.

Furthermore, $ph_{n-1} = h_{n+1}$ [by definition]

Since p is continuous $\lim_{n\to\infty} ph_{n-1} = \lim_{n\to\infty} h_{n+1}$

$$p \lim_{n\to\infty} h_{n-1} = \lim_{n\to\infty} h_{n+1}$$

$$py_0 = y_0$$

Similarly $qy_0 = y_0$.

Now y_0 is a common fixed point of p& q.

If there is another fixed point y_1 of p & q with $y_0 \neq y_1$, then

$$\chi\big(d(y_0, y_1)\big) = \chi\big(d(qy_0, py_1)\big) = \chi\big(d(qy_0, pqy_1)\big)$$

$$\leq \eta(d(y_0, py_1)\chi(w(y_0, py_1))$$

$$\leq \eta(d(y_0, y_1)\chi(w(y_0, y_1)) \qquad (2.21)$$

Where

$$w(y_0, y_1) = w(y_0, py_1) = max\left\{\frac{d(y_0, qy_0)d(py_1, qpy_1)}{d(y_0, py_1)}, d(y_0, py_1)\right\}$$

$$= max\left\{\frac{d(y_0, y_0)d(py_1, y_1)}{d(y_0, y_1)}, d(y_0, y_1)\right\}$$

$$= max\{0, d(y_0, y_1)\} = d(y_0, y_1)$$

From (2.21) & (2.22) we get (2.22)

$$\chi\big(d(y_0, y_1)\big) < \chi\big(d(y_0, y_1)\big)$$

Which is a contradiction to our assumption. Hence y_0 is the unique common fixed point of p & q in H

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