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A Fixed Point Theorem on complete Partially Ordered Cone 2-Metric Spaces using Generalized control Function

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Abstract:
The aim of the Research article is to prove a theorem of unique fixed point using the concept of generalized control function in partial cone 2- metric spaces applying the scheme called c-sequences. We also establish that the fixed point is unique. The presented theorem extends and unifies various fixed point results. To justify the result, few lemmas are used.

Keywords: *cone 2-metric spaces, altering distances (control function) and invertible element.*

1. Introduction

Despite its simplicity, the theory functional Analysis achieves a healthy role due to its broad applications to nonlinear sciences. As far as we know the first significant result is Banach Contraction Principle. It is being served as a very popular tool for solving existence problems in many branches in Mathematical analysis. The term called cone metric spaces which is a generalization of the a particular type of metric spaces called as classical metric space was popularized by Huang and Zhang . 2-metric spaces were established by Gahler. Sharma et.al. Enquired about the occurence and uniqueness of the fixed points of a group of mappings in 2-metric spaces. By joining both the concepts of 2-metric spaces and cone metric spaces, Singh et.al found a new space, called cone 2- metric space and showed several fixed point theorems of a contractive mappings on cone 2-metric spaces. Altering Distance function, also known as control function was introduced by Khan et.al in 1984.

2. Preliminaries:

A multiplication operation is defined, in a real Banach space B obeying the following conditions. For all $x_1,x_2,x_3 \in B, a \in R$:

(i) $(x_1x_2)x_3 = x_1(x_2x_3)$

(ii) $x_1(x_2+x_3) = x_1x_2 + x_1x_3$

(iii) $\alpha(x_1x_2) = (\alpha x_1)x_2 = x_1(\alpha x_2)$

(iv) $\|x_1x_2\| \le \|x_1\|\|x_2\|$

2.1 Definition: Invertible element: [20]
An element $x \in A$ is known to be an invertible if there is an inverse element $y \in A$ such that xy=yx=e, the unit element. The inverse of x is denoted by $x^{-1.}$

2.2 Definition: cone metric spaces: [9]
Assume that X be a non-void set. Consider the mapping d: XxX→ E satisfies

i. 0≤ $d(x,y)$ for all x,y∈ X and d(x,y)=0 iff x=y
ii. $d(x,y) = d(y,x)$ for all x,y∈ X.

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iii. $d(x,y) \leq d(x,z) + d(z,y)$ for all x,y,$z \in X$.

then d is called cone metric on X and (X,d) is known as a cone metric space.

2.3 Definition: Altering Distances:[17]

Let ψ be a function for which,a mapping $\psi: [0,+\infty)^n \rightarrow [0,+\infty)$ is called to be a generalized altering distance or control function if:

(i) ψ is continuous
(ii) ψ *increases* in each of its variables.
(iii) $\psi(t_1, t_2, \ldots . t_n) = 0$ iff $t_{1=}t_2 = \cdots = t_n = 0$

The following Lemmas are helpful in proving the main result.

2.4:Lemma: [20]
Let A be Banach Algebra along with an unit element e and assume $u \in A$. If the spectral Radius r(u) of u < 1,

$$r(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|u^n\|^{\frac{1}{n}} < 1,$$

At that point e-u is invertible. In fact $(e-u-1=i=0\infty ui$

2.5:Lemma[20]
Let A be a Banach Algebra and let x_1,$x_2$be the vectors in A. If $x_1$andx_2 commute, then the following conditions are true.

(i) r $(x_1x_2) \leq r(x_1)r(x_2)$
(ii) r$(x_1+x_2) \leq r(x_1) + r(x_2)$
(iii) $|r(x_1) - r(x_2)| \leq r(x_1-x_2)$.

2.6:Lemma:[20]
Let A be a Banach Algebra and let k be the vector in A. If $0 \leq r(k) < 1,$ then r$((e-k)^{-1})\leq (1-r(k))^{-1}$

2.7:Definition: 2-Metric space:[5]
Let X be a set which is non-empty . Suppose that the mapping p:XxXxX→ R^{+}satisfies:

(i) for any two of distinct points x_1,$x_2 \in X$, there is a point $x_3 \in X$ such that p(x_1,x_2, x_3)$\neq 0$
(ii) p(x_1,x_2, x_3)= 0⟺atleast two of x_1,$x_2, x_3$are equal.
(iii) p(x_1,x_2, x_3)= $p\big(d(x_1, x_2, x_3)\big)$for all,$x_1$,$x_2, x_3 \in X$ and for all permutations and combinations d(x_1,x_2, x_3) of x,y,zx_1,x_2, x_3
(iv) p(x_1,x_2, x_3)≤ p(x_1, x_2, w) + p(x_1, w, x_3) + p$(w, x_2, x_3)$for all x_1,x_2, x_3,w $\in X$

Then p is called the 2-metric on X and (X,p) is called a 2-metric space.

2.8:Definition: Cone 2-metric space:[21]
Assume X is a non-void set. Suppose that the mapping p: XxXxX→ A satisfies:

(i) for every pair of distinct points x_1,$x_2 \in X$, there exists a point $x_3 \in X$ such that p(x_1,x_2, x_3)$\neq \theta$
(ii) $\theta \preccurlyeq$p(x_1,x_2, x_3) for all x_1,$x_2, x_3 \in X$and p(x_1,x_2, x_3)= θiff atleast two of x_1,$x_2, x_3$are equal.
(iii) $p(x_1$,$x_2, x_3) = p\big(d(x_1, x_2, x_3)\big)$for all$x_1$,$x_2, x_3 \in X$ and for all permutations d(x_1,x_2, x_3) of x_1,x_2, x_3
(iv) p(x_1,x_2, x_3)≤ p(x_1, x_2, w) + p(x_1, w, x_3) + p$(w, x_2, x_3)$for all x_1,x_2, x_3,w $\in X$

Then p is called the cone 2-metric on X and (X,p) is known to be a cone 2-metric space over the A.

2.9:Definition:Solid Cone:[13]
A subset C of A is called a cone of A if:

(i) C is non-empty closed and $\{\theta, e\} \subset C$
(ii) $\alpha P + \beta P \subset C$ for all positive real numbers α, β
(iii) $C^2 = CC \subset C$
(iv) C∩ $(-C)$={θ}, where θ is the null element of the Banach algebra A.

For a given cone C $\subset A$, we characterize a partial ordering ≼ concerning C by x≼ y iff y-x∈ C.

x≺y implies x≼ y and x≠ y,

then$x \ll y$ valid for y-x $\in intC$, $where$ int C is the interior of C.

If int of C≠ $\emptyset$, $then$ C is called a solid cone.

2.10:Lemma:[20]

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If E is a Banach space which is also real with Cas a solid cone and if $\theta \preccurlyeq u \ll c$ for each $\theta \ll c$, then u= θ.

2.11:Lemma:[20]

If A will be a real Banach space with a solid cone C and if $\|x_n\| \to 0(n \to \infty)$

At that point for any $\theta \ll c,$ there existsN $\in \mathcal{N}$,to such an extent that, for any n$> N$, we have $x_n \ll c$.

2.12:Definition: c-sequence:[4],[15]

Ler C be a solid cone in A which is a Banach space and a sequence $\{x_n\} \subset C$ is a c-sequence if for each there existsn $\in \mathcal{N}$, such that $x_n \ll c$for all n$> N$.

2.13:Lemma:[20]

Let C be a solid cone in a Banach space A. Two sequences $\{x_n\}$ and $\{y_n\}$be c- sequences in C. If $\{x_n\}$ and $\{y_n\}$ be c-sequences and $\alpha \; and \; \beta > 0$ then $\{\alpha x_n + \beta y_n\}$is a c- sequence.

2.14:Lemma:[20]

Let us have C be a solid cone in a Banach space A and let $\{x_n\}$be sequence in C. Suppose that a$\in C$ is an arbitrary vector and$\{x_n\}$be any c-sequence in C. Then $\{ax_n\}$ is also a c-sequence.

2.15:Proposition [21]

Let us take (X,p) to be a cone 2- metric space which is complete over the Banach Algebra A and let C be the solid cone in A. Let $\{x_n\}$be a sequence in X. If $\{x_n\}$ converges to x $\in X,$ then the following hold:

(i)$\{p(x_n, x, a)\}$is a c- sequence for all a$\in X$.
(ii) For any d $\in \mathcal{N}, \{p(x_n, x_{n+p}, a)\}$is a c-sequence for all a$\in X$.

2.16:Proposition:[21]

Let us consider a solid cone C in the Banach Algebra A and let x be a vector in A. Assume that k$\in C$ is an arbitrarily given vector and $x \ll c$ for any $\theta \ll c$, at that point we have $kx \ll c$ for any $\theta \ll c$.

Proof:

Let us c$\gg \theta$, then $\frac{c}{m} \gg \theta$ for all m$\in \mathcal{N}$.It is clear that $x \preccurlyeq \frac{c}{m}$ for all m$\in \mathcal{N}$.So kx$\preccurlyeq \frac{kc}{m}$. Since $\frac{kc}{m} \to \theta$ as m$\to \infty$, there existsM$\in \mathcal{N}$ such that kx$\preccurlyeq \frac{kc}{m} \ll c$ when m$> M$.

MAIN RESULT:

Let $(X, p, \preccurlyeq)$ be a partially ordered complete Cone-2 Metric space over the B and C be any solid cone. Let $\{F_i\}$,i=1 to ∞ be a family of mappings from X to itself. Suppose that there exists a positive integer sequence $\{m_i\}$,i=1 to ∞ so that for all positive integer i,j and for all x,y,z$\in X$, then ,

$$\begin{aligned}\psi_1\left(p\left(F_i^{\,m_i}x, F_j^{\,m_j}y, z\right)\right) &\preccurlyeq \psi_1\left[a\left(p(x, F_i^{\,m_i}x, z)\right) + b\left(p\left(y, F_j^{\,m_j}y, z\right)\right) + c\left(p(x, y, z)\right)\right] \\ &- \;\psi_2\left[a\left(p(x, F_i^{\,m_i}x, z)\right), b\left(p\left(y, F_j^{\,m_j}y, z\right)\right), c\left(p(x, y, z)\right)\right]\end{aligned}$$

Where a, b, c $\in C$ with $r(a) + r(b) + r(c) < 1$ and $\psi_1, \psi_2$are generalized altering distance functions with $\psi_1(s)$= $\psi_1(s, s, s, s \ldots\ldots)$, then the sequence $\{F_i\}$,i=1 to ∞ have a unique fixed point in X.

Proof:

Let$t_i = \{F_i\}$, i=1 to ∞. Then for all i and j, we have

$$\begin{aligned}\psi_1\left(p\left(t_i(x), t_j(y), z\right)\right) &\preccurlyeq \psi_1\left[a\left(p\left(x, t_j(x), z\right)\right) + b\left(p\left(y, t_j(y), z\right)\right) + c\left(p(x, y, z)\right)\right] \\ &- \psi_2\left[a\left(p\left(x, t_j(x), z\right)\right), b\left(p\left(y, t_j(y), z\right)\right), c\left(p(x, y, z)\right)\right]\end{aligned}$$

Let $x_0 \in X$ and set a sequence $x_n = t_n(x_{n-1})$, n$\geq$ 1. Then

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$$\begin{aligned}\psi_1[(p(x_{n+1},x_n,z))] &= \psi_1[(p(t_{n+1}(x_n),t_n(x_{n-1}),z))]\\ &\preccurlyeq \psi_1[a(p(x_n,x_{n+1},z)) + b(p(x_{n-1},x_n,z))\\ &+ c(p(x_n,x_{n-1},z))] - \psi_2[a(p(x_n,x_{n+1},z)),b(p(x_{n-1},x_n,z)),c(p(x_n,x_{n-1},z))]\end{aligned}$$

From the lemma 2.4, we get

$$\psi_1[(p(x_{n+1},x_n,z))(1-a)] \preccurlyeq \psi_1[b(p(x_{n-1},x_n,z)) + c(p(x_n,x_{n-1},z))]$$

$$-\psi_2[a(p(x_n,x_{n+1},z)),b(p(x_{n-1},x_n,z)),c(p(x_n,x_{n-1},z))]$$

$$\psi_1[(p(x_{n+1},x_n,z))] \preccurlyeq$$

$$\left[\psi_1\left(\frac{b+c}{e(1-a)}\right)(p(x_n,x_{n-1},z))\right]$$

$$-\psi_2[a(p(x_n,x_{n+1},z)),b(p(x_{n-1},x_n,z)),c(p(x_n,x_{n-1},z))]$$

$\preccurlyeq \psi_1[k(p(x_n,x_{n-1},z))] - \psi_2\begin{bmatrix}a(p(x_n,x_{n+1},z)),\\ b(p(x_{n-1},x_n,z)),\\ c(p(x_n,x_{n-1},z))\end{bmatrix}$ Where k= $\left(\frac{b+c}{e(1-a)}\right)$

$$\preccurlyeq \psi_1[k(p(x_n,x_{n-1},z))] - \psi_2\begin{bmatrix}a(p(x_n,x_{n+1},z)),\\ b(p(x_{n-1},x_n,z)),\\ c(p(x_n,x_{n-1},z))\end{bmatrix}$$

For the convergence, we have

$$\preccurlyeq \psi_1[k^2(p(x_n,x_{n-1},z))] - \psi_2\begin{bmatrix}a(p(x_n,x_{n+1},z)),\\ b(p(x_{n-1},x_n,z)),\\ c(p(x_n,x_{n-1},z))\end{bmatrix}$$

$$\preccurlyeq \psi_1[k^n(p(x_n,x_{n-1},z))] - \psi_2\begin{bmatrix}a(p(x_n,x_{n+1},z)),\\ b(p(x_{n-1},x_n,z)),\\ c(p(x_n,x_{n-1},z))\end{bmatrix}$$

$[(p(x_n,x_{n-1},x_l))] \preccurlyeq (p(x_{n-1},x_{n-2},x_l))$

$$\preccurlyeq\ k^{n-l-1}[\psi_1(p(x_{l+1},x_l,x_l))]$$

$\Rightarrow p(x_n,x_{n-1},x_l) = \theta$for all l<n. Then for n>m,

$\psi_1[(p(x_n,x_m,z))] \preccurlyeq\ \psi_1[(p(x_n,x_m,x_{n-1})) + (p(x_n,x_{n-1},z)) + (p(x_{n-1},x_m,z))]$

$-\ \psi_2[(p(x_n,x_m,x_{n-1})) + (p(x_n,x_{n-1},z)) + (p(x_{n-1},x_m,z)),]$

$$\preccurlyeq \; k^{n-1}\Big\{\psi_1\big[\big(p(x_1,x_0,z)\big)+\big(p(x_{n-1},x_m,x_{n-2})\big)+\big(p(x_{n-1},x_{n-2},z)\big) + \big(p(x_{n-2},x_m,z)\big)\big] - \psi_2\Big[\Big(\big(p(x_1,x_0,z)\big)+\big(p(x_{n-1},x_m,x_{n-2})\big)+\big(p(x_{n-1},x_{n-2},z)\big) + \big(p(x_{n-2},x_m,z)\big)\Big)\Big]\Big\}$$

$$\preccurlyeq (k^{n-1}+k^{n-2})\big[\big(p(x_1,x_0,z)\big)+\big(p(x_{n-2},x_m,z)\big)\big]$$

Continuing like this we get,

$\preccurlyeq (k^{n-1}+k^{n-2}+\cdots k^{m+1})\big[\big(p(x_1,x_0,z)\big)+\big(p(x_{m+1},x_m,z)\big)\big]$

$\preccurlyeq (k^{n-1}+k^{n-2}+\cdots k^{m+1}+k^m)\big(p(x_1,x_0,z)\big)$

$=(e+k+\cdots k^{n-m+1})k^m p(x_1,x_0,z)$

From lemma 2.2 &2.3, we have

$$r(k) = r[(e-a)^{-1}(b+c)] \preccurlyeq r((e-a)^{-1}r(b+c)$$

$$\preccurlyeq \frac{r(b+c)}{1-r(a)} \preccurlyeq \frac{r(b)+r(c)}{1-r(a)} < 1$$

So from lemma 2.5 and from the concept of spectral radius, we have

$\|(e-r)^{-1}r^m p(x_1,x_0,a)\| \to 0 \; as \; n\to\infty$. It follows that for any $c \,\epsilon A$ with $\theta \ll c$

Therefore $\{x_n\}$ is a Cauchy sequence in X.

We know that X is complete. Then there exists x$\in X$ such that $x_n \to x$as n$\to \infty$

To claim x is the fixed point of F_i:
Have x be the fixed point, Say. i.e.to prove that$t_n(x)=x$. Actually

$$\psi_1\big[\big(p(t_n(x),x,z)\big)\big] \preccurlyeq \psi_1\big[\big(p(t_n(x),x,t_{m+1}(x_m)\big)+p\big((t_n(x),t_{m+1}(x_m),z\big)+p\big((t_{m+1}(x_m),x,z\big)\big]$$

$$-\psi_2\big[\big(p(t_n(x),x,t_{m+1}(x_m)\big)+p\big((t_n(x),t_{m+1}(x_m),z\big)+p\big((t_{m+1}(x_m),x,z\big)\big]$$

$$\preccurlyeq \psi_1\big[\big(p(x_{m+1},x,z)\big)+a\,(p(x,t_n(x),z)+b\,(p(x_m,t_{m+1}(x_m),x)+c\big(p(x,x_m,x)\big) + a(p(x_m,t_{m+1}(x_m),z)+b(p(x,t_n(x),z)+c\big(p(x_m,x,z)\big) - \psi_2[\big(p(x_{m+1},x,z)\big),a(p(x,t_n(x),z),b(p(x_m,t_{m+1}(x_m),x),c\big(p(x,x_m,x)\big) + a(p(x_m,t_{m+1}(x_m),z),b\,p(x,t_n(x),z),c\big(p(x_m,x,z)\big)\big] \Rightarrow (e-b)\psi_1\big[\big(p(t_n(x),x,z)\big)\big] \preccurlyeq \psi_1\big[\big(p(x_{m+1},x,z)\big)+b\big(p(x_m,x_{m+1},x)\big)+a\big((p(x_m,x_{m+1},z)\big)+c\big(p((x_m,x,z)\big)\big]$$

Therefore from lemma 2.13 &2.14 and proposition 2.15, then

$$(e-b)\psi_1\big(p(t_n(x),x,z)\big) \preccurlyeq q_m,$$

where$\{q_m\}$ is C-sequece $\in$ C.

Now we have $(e-k_2)\big(p(t_n(x),x,z)\big) \ll c$, for any $c \gg \theta$, with the usage of proposition3.2,

$\theta \preccurlyeq \big(p(t_n(x),x,z)\big) \ll c \Rightarrow t_n(x)$is a c-sequence as $e-k_2$ is invertible, $\gg \theta$.

By lemma 2.11, $\left(p(t_n(x), x, z)\right)$ =**c,** for n∈ N. Therefore $t_n(x) = x$ for anyn∈ N.

To Prove the Uniqueness:

Suppose there exists an alternate fixed point say, $y \in t_n(x) \in X$

Then,

$$\psi_1\big(p(x,y,z)\big) \preccurlyeq \psi_1\big[a\big(p(x,x,z)\big) + b\big(p(y,y,z)\big) + c\big(p(x,y,z)\big)\big] - \psi_2[p(x,x,z), p(y,y,z), p(x,y,z)]$$

$(e-b)\psi_1\big(p(x,y,z)\big) \preccurlyeq \theta, \Rightarrow p(x,y,z) = \theta$,for z∈X

$(e-b)$is invertible,then we have x=y ⇒fixed point is unique.

Conclusion:

This research article is mainly focused on cone 2-metric spaces and altering Distances alias control function. The applications of cone 2-metric is in multiple number of branches Mathematics, few of which are Integral equations, Initial Value Problems, Dynamic Programming etc.

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