TEST Engineering & Management

September-October 2019
ISSN: 0193-4120 Page No. 223-227

A Convolution Approach to Certain Subclasses of Multivalent Functions Related to Complex Order

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Article Info
Volume 83
Page Number: 223 - 227
Publication Issue:
September-October 2019

Article History
Article Received : *3 January 2019*
Revised: *25 March2019*
Accepted: *28 July2019*
Publication: *20 October 2019*

ABSTRACT

Let $V^{b}_{\lambda,p,\mu}[A,B]$ denote the class of functions $f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n}z^{p+n}$

analytic in $u=\{z:|z|<1\}$, such that $p+\frac{1}{b}\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}}-p\right\}=p(1-\mu)+p\mu\left\{\frac{1+Az}{1+Bz}\right\}, z\in u,$

where $-1\le B<A\le 1, 0<\mu\le 1, \lambda>-p$ and b is any non-zero complex number. In this paper, we investigate certain properties of the above-mentioned class.

1. INTRODUCTION

Let $\mathcal{A}_p$ denote the class of functions $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}, p$ is a positive integer, which are analytic in the unit disc $u=\{z:|z|<1\}$. If f and g are any two functions in the class $\mathcal{A}_p$ such that $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ and $g(z)=z^p+\sum_{n=1}^{\infty}b_{p+n}z^{p+n}$, then the convolution or Hadamard product of f and g, denoted by $f*g$, is defined by the power series

$$(f*g)(z)=z^p+\sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^{p+n}$$

Let

$$D^{\lambda+p-1}f(z)=\frac{z^p\left(z^{\lambda-1}f(z)\right)^{(\lambda+p-1)}}{(\lambda+p-1)!}, \lambda>-p$$

Then, following Al-Amiri [1], we shall1 refer to $D^{\lambda+p-1}, f(z)$ as the $(\lambda+p-1)^{th}$ order Rugcheweyh derivative of the function f. It ia easy to observe that

$$D^{\lambda+p-1}f(z)=\frac{z^p}{(1-z)^{\lambda+p}}*f(z).$$

Goel and Sohi [5] studied the class $\mathcal{S}_{\lambda,p}(\beta)$ of those functions of $\mathcal{A}_p$ which satisfy

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$$\text{Re}\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}}\right\} > p\beta,\ z\in u \qquad (1.1)$$

where $0\le\beta<1$. They showed that the functions in the class $\mathcal{S}_{\lambda,p}(\beta)$ are p-valent in u. As usual for other class of p-valent functions, β may be called the order of function in the class $\mathcal{S}_{\lambda,p}(\beta)$, Aouf ([2], [3]) and Nasr and Aouf ([9], [10], [11], [12]) have introduced some classes of univalent and p-valent functions of complex order. But no one has, so far, introduced a class of functions of complex order in this direction defined by Convolution. Considering this natural problem, we now introduce a class $V^b_{\lambda,p,\mu}[A,B]$ as defined below:

A function of f of $\mathcal{A}_p$ belongs to the class $V^b_{\lambda,p,\mu}[A,B]$ if and only if there exist a function $W(z)$ analytic in u and satisfying $W(o)=0$ and $|W(z)|<1$ for $z\in u$, such that

$$p+\frac{1}{b}\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}}-p\right\}=p(1-\mu)+p\mu\left[\frac{1+AW(z)}{1+BW(z)}\right],\ z\in u \qquad (1.2)$$

where $-1\le B<A\le 1, 0<\mu\le 1, \lambda>-p$ and b is any non-zero complex number.

It is easy to see that the condition (1.2) is equivalent to

$$\left|\frac{\frac{\left(D^{\lambda+p-1}f(z)\right)'-p}{z^{p-1}}}{\mu(A-B)bp-B\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}}-p\right\}}\right|<1,\ z\in u. \qquad (1.3)$$

By giving specific value to λ,μ,b,A and B in (1.3), we obtain the following subclasses studied by various researchers in earlier works:

(i) For $b=\cos\ \delta e^{-i\delta}$, we obtain the class of functions $f(z)$ satisfying the condition

$$\left|\frac{e^{i\delta}\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}}-p\right\}}{\mu(A-B)p\cos\ \delta-Be^{i\delta}\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}}-p\right\}}\right|<1,\ z\in u.$$

where $\delta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, studied by Shukla and Chaudhary [13].

(ii) For $\mu=1$ and $b=1$, we obtain the class of functions $f(z)$ satisfying the condition

$$\left|\frac{\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}}-p\right\}}{(A-B)p-B\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}}-p\right\}}\right|<1,\ z\in u$$

studied by Kumar and Shukla [8].

(iii) For $\mu=1, A=(1-2\beta), B=-1$ and $b=1$, we obtain the class of functions $f(z)$ satisfying the condition (1.1), studied by Goel and Sohi [5].

(iv) For $\mu=1, b=1$ and $\lambda=1-p$, we obtain the class of functions $f(z)$ satisfying the condition

$$\left|\frac{\frac{f'(z)}{z^{p-1}}-p}{(A-B)p-B\left\{\frac{f'(z)}{z^{p-1}}-p\right\}}\right|<1, z\in u$$

studied by Chen [4].

Thus the study of the class $V^b_{\lambda,p,\mu}[A,B]$ provides a unified approach for the classes considered by Shukla and Chaudhary [13], Kumar and Shukla [8], Goel and Sohi [5], and Chen [4].

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In the present paper, firstly, we obtain the basic inclusion relation

$$V^b_{\lambda+1,p,\mu}[a,B] \subset V^b_{\lambda,p,\mu}[A,B].$$

Then we obtain coefficient estimate, sufficient condition in terms of coefficients distortion theorem and maximization of $|a_{p+2} - \beta a^2_{p+1}|$ over the class $V^b_{\lambda,p,\mu}[A,B]$. In the last, we show that the class $V^b_{\lambda,p,\mu}[A,B]$ is closed under "arithmetic mean".

2. PRELIMINARY LEMMAS

We state lemma 2.1 due to Jack [6], lemma 2.2 due to Keogh and Merkes [7] and prove a lemma 2.3 that are needed in section 3 .

Lemma 2.1. If the function W is analytic for $|z| \le r < 1, W(o) = 0$ and $|W(z_0)| = \max_{|z|=r}|W(z)|$, then

$$z_0 W'(z_0) = \xi W(z_0)$$

where ξ is a real number such that $\xi \ge 1$.

Lemma 2.2. Let $W(z) = \sum_{k=1}^{\infty} c_k z^k$ be analytic with $|W(z)| < 1$ in u. If d is a complex number, then

$$|c_2 - dc_1^2| \le \text{m} \ \{1, |d|\}.$$

Equality may be attained with the functions $W(z) = z^2, W(z) = z$.

Lemma 2.3. A function $f(z)$ of $\mathcal{A}_p$ belongs to the class $V^b_{\lambda,p,\mu}[A,B], -1 < B < A \le 1$, if and only if

$$|G_p(z) - m| < M, \ z \in u \qquad (2.1)$$

where

$$G_p(z) = p + \frac{1}{b}\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}} - p\right\} \qquad (2.2)$$

$$m = p - \frac{\mu(A-B)Bp}{1-B^2}$$

and

$$M = \frac{\mu(A-B)p}{1-B^2}.$$

Proof. Suppose that $f(z) \in V^b_{\lambda,p,\mu}[A,B]$. Then from (1.2) we have

$$G_p(z) = p\left[\frac{1+(\mu(A-B)+B)W(z)}{1+BW(z)}\right]$$

where $G_p(z)$ is defined by (2.2).

Therefore

$$G_p(z) - m = \frac{(p-m) + [\{B+\mu(A-B)\}p - Bm]W(z)}{1+BW(z)}$$

$$= M\left[\frac{B+W(z)}{1+BW(z)}\right] = Mh(z) \qquad (2.3)$$

It is clear that the function $h(z)$ satisfies $|h(z)| < 1$. Hence (2.1) follows from (2.3). Conversely, suppose that the inequality (2.1) holds. Then

$$\left|\frac{G_p(z)}{M} - \frac{m}{M}\right| < 1.$$

Let

$$g(z) = \frac{G_p(z)}{M} - \frac{m}{M}$$

and

$$W(z) = \frac{g(z)-g(0)}{1-g(0)g(z)} = \frac{\{G_p(z)-p\}}{\mu(A-B)p - B\{G_p(z)-p\}} \qquad (2.4)$$

Clearly $W(0) = 0$ and $|W(z)| < 1$, Rearranging (2.4) we arrive at (1.3). Hence $f(z) \in V^b_{\lambda,p,\mu}\mu, [A,B]$

Note. The condition (2.1) can also be written as

$$\left|\frac{(1-B)\left(G_p(z)-p\right)+\mu(A-B)p}{\mu(A-B)p}-\frac{1}{1+B}\right|<\frac{1}{1+B},\ z\in u.$$

Now as $B\to -1$, the above condition reduces to

$$\text{Re}\left[G_p(z)\right]>\frac{1}{2}[2-\mu(1+A)]p,\ z\in u,$$

which is equivalent to (1.3), when $B=-1$. Thus including the limiting case $B\to -1$, the results proved with the help of above lemma will hold for $-1\leq B<A\leq 1$.

3. MAIN RESULTS

The proof of each of the following theorems runs parallel to that of the corresponding assertion made by Shukla and Chaudhary [13] in the special case $b=\cos\delta e^{-i\delta}$, and we omit the details involved.

Theorem 3.1. Let λ_0 be any integer such that $\lambda_0>\lambda$. Then

$$V^b_{\lambda_0,p,\mu}[A,B]\subset V^b_{\lambda,p,\mu}[A,B].$$

Theorem 3.2. If $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ belongs to the class $V^b_{\lambda,p,\mu}[A,B]$, then

$$\left|a_{p+n}\right|\leq\frac{\mu(A-B)p|b|}{(p+n)\alpha(\lambda,n)},\ n=1,2,\ldots \quad (3.1)$$

where

$$\alpha(\lambda,n)=\begin{bmatrix}\lambda+p+n-1\\ \lambda+p-1\end{bmatrix} \quad (3.2)$$

The inequality (3.1) is sharp.

Theorem 3.3. Let $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ be analytic in u.

If

$$\sum_{n=1}^{\infty}(1-B)\alpha(\lambda,n)(p+n)\left|a_{p+n}\right|\leq\mu(A-B)p|b|, \quad (3.3)$$

where $\alpha(\lambda,n)$ is defined by (3.2), then $f\in V^b_{\lambda,p,\mu}[A,B]$. The inequality (3.3) is sharp. Further, the converse need not be true.

Theorem 3.4. If $f\in V^b_{\lambda,p,\mu}[A,B]$, then

$$\frac{p(1-B^2r^2)-p\mu Br^2(A-B)\,\text{Re}(b)+\mu(A-B)p|b|r}{1-B^2r^2}\leq\text{Re}\left\{\frac{\left(D^{\lambda+p-1}f(z)\right)'}{z^{p-1}}\right\}$$
$$\leq\frac{p(1-B^2r^2)-p\mu Br^2(A-B)\,\text{Re}(b)+\mu(A-B)p|b|r}{1-B^2r^2}$$

These inequalities are sharp.

Theorem 3.5. If $f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}$ belongs to the class $V^b_{\lambda,p\mu}[A,B]$, then for any complex number β, we obtain

$$\left|a_{p+2}-\beta a_{p+1}^2\right|\leq\frac{\mu(A-B)p|b|}{\alpha(\lambda,2)(p+2)}\text{m}\ \{1,|d|\}$$

where

$$d=\frac{\mu(A-B)bp\beta\alpha(\lambda,2)(p+2)+B\{\alpha(\lambda,1)\}^2(p+1)^2}{\{\alpha(\lambda,1)\}^2(p+1)^2}.$$

The result is sharp.

Theorem 3.6. Let c be a real number such that $c>-p$. If $f(z)\in V^b_{\lambda,p,\mu}[A,B]$, then the function $F(z)$ defined by

$$F(z)=\frac{(c+p)}{z^c}\int_0^z t^{c-1}f(t)dt$$

also belongs to the $V^b_{\lambda,p,\mu}[A,B]$.

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Theorem 3.7. If the function $f(z)$ and $g(z)$ belong to $V_{\lambda,p,\mu}^{b}[A,B]$ and $0 \leq s \leq 1$, then the function $F(z)$ defined by

$$F(z) = sf(z) + (1-s)g(z)$$

also belongs to $V_{\lambda,p,\mu}^{b}[A,B]$.

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