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Mathematical Modeling of Nonlinear Roll Motion of Ships using Homotopy Perturbation Method

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Abstract:
Mathematical modelling of the nonlinear roll motion of a ship is discussed. The model is based on second- order nonlinear equations containing a nonlinear term related to damping and restoring moment. *Analytical solution of the nonlinear equations is obtained using a modified homotopy perturbation method.* For normal or irregular waves, roll angle, roll velocity, and moment of damping and restoring for the particular ship are also provided. A comparison of the theoretical approximation obtained via the proposed approach shows strong agreement with the numerical results.

Keywords: *Mathematical modelling, Numerical simulation, Non-linear rolling, Peak amplitude, Homotopy perturbation method.*

I. INTRODUCTION

For every ship, protection from capsizing is of great importance. A great effort is being made to develop a basic ship safety evaluation methodology that is applicable during the design process and ship operation. One of the leading causes of a ship capsizing in waves is the loss of stability due to nonlinear roll motion .

The roll motion of ships is described by second order differential equation with both linear and non-linear damping coefficients. Typically the linear method is used for low rolling amplitudes, and the spectral analysis describes the problem in the frequency domain. Large-amplitude ship movements are expected to result in strongly nonlinear, even chaotic behavior. In this case, the non-linear rolling formula, which could be set for regular or irregular waves, aims at foreseeing the ship's non-linear response. The ship rolling problem could be studied in the frequency and time domain . The analysis in the time domain is more acceptable for this problem than the report in the frequency domain. A disadvantage is a need for conducting a significant number of realizations to determine the probability of capsizing. The advancement of numerical simulation and probabilistic analysis is enabling the development of instructions for ship handling based on risk.

Numerical methods for the simulation of ship motion are built in parallel with the growing computing power. The mathematical techniques used in the literature vary in complexity and ability to compensate for a different phenomenon of flow (e.g. two-phase flow, turbulence, ship motion) [1]. An overview of the empirical methods to predict roll damping is summarized by Flazarano et al. [2]. Graham et al. [3] discussed the viscous damping prediction of large floating bodies in waves. Seah et al. [4] have coupled a vortex method for simulating the roll decline of a floating cylinder with rigid body design. Several scientists studied the effects of a moving ship's nonlinear recovery moments. Due to waves and manoeuvring, Koskinen [5] discussed the numerical simulation of ship motion. Ibrahim [6] presented the modelling of ship roll dynamics and its coupling with heave and pitch. The oscillation of the roll is one of the most important movements that can result in the boat being capsized. Ibrahim [6] articulated the ships ' response through a linear formula for small angles of roll motions. Kari Unneland and Fossen carried out the theoretical studies on low-order potential damping models for surface vessels [7]. Perez and Fossen [8] presented a detailed simulation model of the naval coastal patrol vessel. Wu [9] obtained an exact solution for the design of a ship hull's heave and pitch movements. Thuhad [10] investigated the coupled heave and pitch motions of a non-uniform hull on a still surface.

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In this paper, the nonlinear model of roll motion of the ship in the time domain has been analyzed by using a modified homotopy perturbation method. Numerical simulations are performed to investigate the characteristics of the movements of ships and to validate the proposed mathematical model. The damping and restoring moments are reported, which are the essential parameters of ship dynamics.

Nomenclature

Symbols	Definitions	Units
θ	Roll angle	*rad*
t	Time	*Sec*
I	Mass inertia moment	$kg\ \ m^2$
B	Moment of non-linear damping	$N.m$
C	Restoring moment	$N.m$
M	Wave a moment of excitement	$N.m$
d_1	Relative coefficient of damping	*None*
$k_{1,}k_3,k_5$	Relative restoring coefficient	*None*
ω	Frequency	*Hertz*
a	Excitation amplitude	m

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Using Newton's law of motion, the roll motion of the ship can be described as the following second-order ordinary differential equation.

$$A\ddot{x} + B\dot{x} + Cx + f(t) = 0 \tag{1}$$

In the above equation, capital letters A, B and C represent the coefficient of inertia, damping and restoring terms. The first-term represents the forces of inertia, and the second and third-term describes the moment of damping and regeneration. Equation (1) can be reduced to a single differential equation with one degree of freedom as follows:

$$I\ddot{\theta} + B\dot{\theta} + C\theta = M_\theta(t) \tag{2}$$

The roll angle θ is the only independent variable. The coefficients of the equation are the moment of inertia, the roll damping, and the restoring moment. The roll damping, and the restoring moment are nonlinear function of roll angle. Fig.1 represents general roll damping prediction model.

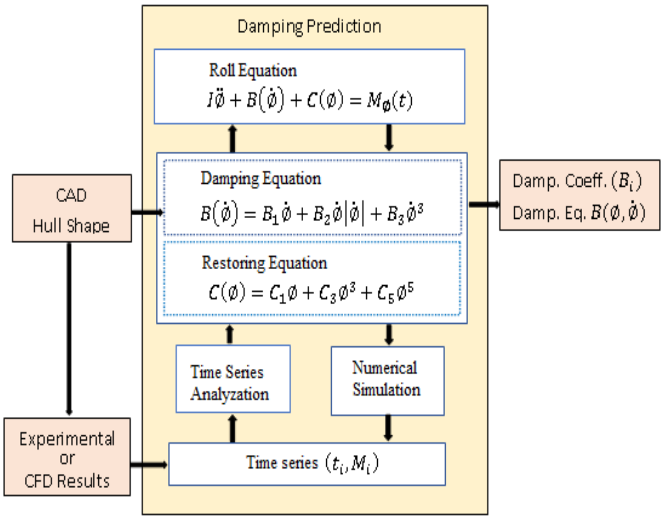

Fig. 1.Design of the roll damping prediction system development.

3. ANALYTICAL EXPRESSION OF ROLL ANGLE USING NEW HOMOTOPY PERTURBATION METHOD

The homotopy perturbation method is a powerful and efficient method for finding the solution of nonlinear differential equations (see Appendix A). The HPM technique was first introduced by He [12,13] in 1998. In recent years, HPM has been employed by many authors to find reliable solutions for many kinds of linear and nonlinear equations arising in physical and chemical and engineering sciences [13-17].

3.1. Linear damping and nonlinear restoring moment with wave exciting moment

The following second-order, non-linear differential equation describes the roll motion for a damaged ship [11].

$$\ddot{\theta}(t) + d_1\dot{\theta}(t) + k_1\theta(t) + k_3\theta^3(t) + k_5\theta^5(t) - a\cos(\omega t) = 0, \tag{3}$$

where d_1 is relative damping coefficients, k_1, k_3 and k_5 are relative restoring coefficients, a is excitation amplitude and ω is the frequency. The initial and boundary conditions are

$$\theta = l \text{ for } t = 0. \tag{4}$$

$$\frac{d\theta}{dt} = 0 \text{ for } t = 0. \tag{5}$$

Using homotopy perturbation method method (Appendix B), the roll angle can be obtained as follows:

$$\theta(t)=e^{\frac{-d_1 t}{2}}\left[\left(l+\alpha-\beta\left(k_1-\omega^2\right)\right)\cos\left(\frac{\gamma t}{2}\right)+\left(l+\alpha-\beta\left(k_1+\omega^2\right)\right)\frac{d_1}{\gamma}\sin\left(\frac{\gamma t}{2}\right)\right]-\alpha+\beta\left[d_1\omega\sin\omega t+\left(k_1-\omega^2\right)\cos\omega t\right]$$

(6)

and hence roll velocity is

$$\frac{d\theta}{dt}=e^{\frac{-d_1 t}{2}}\left[-\beta d_1\omega^2\cos\left(\frac{\gamma t}{2}\right)+\left(\left(l+\alpha-\beta\left(k_1+\omega^2\right)\right)\left(-\frac{d_1^2}{2\gamma}-\frac{\gamma}{2}\right)\right)\sin\left(\frac{\gamma t}{2}\right)\right]+\beta\left[d_1\omega^2\cos\omega t-\omega\left(k_1-\omega^2\right)\sin\omega t\right], \qquad (7)$$

and acceleration is

$$\frac{d^2\theta}{dt^2}=e^{\frac{-d_1 t}{2}}\left[\left(\frac{\beta d_1^2\omega^2}{2}+\left(l+\alpha-\beta\left(k_1+\omega^2\right)\right)\left(\frac{-d_1^2}{4}-\frac{\gamma^2}{4}\right)\right)\cos\left(\frac{\gamma t}{2}\right)+\left(\frac{\beta d_1\omega^2\gamma}{2}+\left(l+\alpha-\beta\left(k_1+\omega^2\right)\right)\left(\frac{d_1^3}{4\gamma}+\frac{d_1\gamma}{4}\right)\right)\sin\left(\frac{\gamma t}{2}\right)\right] \qquad (8)$$
$$+\beta\left[-d_1\omega^3\sin\omega t-\omega^2\left(k_1-\omega^2\right)\cos\omega t\right]$$

where the constants

$$\alpha=\frac{k_3 l^3}{k_1}+\frac{k_5 l^5}{k_1},\ \beta=\frac{a}{d_1\omega^2+\left(k_1-\omega^2\right)^2},\ \gamma=\sqrt{4k_1-d_1^2}\ . \qquad (9)$$

The roll angle and velocity for the experimental value of the parameters

$d_1=0.0126513, k_1=0.67199703, k_3=-0.5392039, k_5=-0.0867972,\ a=0.1,\ \omega=0.1,\ l=0.3$ are obtained as follows:

$$\theta(t)=e^{-0.0063t}\left[0.127\cos(0.8062t)+0.00096\sin(0.8062t)\right]+0.0002\sin(0.1t)+0.1510\cos(0.1t)+0.022. \qquad (10)$$

$$\frac{d\theta}{dt}=e^{-0.0063t}\left[-0.00002\cos(0.8062t)-0.102381\sin(0.8062t)\right]+0.00002\cos(0.1t)-0.0151\sin(0.1t) \qquad (11)$$

Using the roll angle and velocity, restoring moment and damping moment can be obtained as follows:

$$C\theta(t)=k_1\theta(t)+k_3\theta^3(t)+k_5\theta^5(t)=0.671997033\,\theta(t)-0.53920393\,\theta^3(t)-0.0867972\ \theta^5(t) \qquad (12)$$

$$B\dot{\theta}(t)=d_1\dot{\theta}(t)=0.01265913\dot{\theta}(t) \qquad (13)$$

3.2. Linear damping and non-linear restoring moment without wave exciting moment.

The nonlinear equation for the ship without wave exciting moment can be written as follows [11]:

$$\ddot{\theta}(t)+d_1\dot{\theta}(t)+k_1\theta(t)+k_3\theta^3(t)+k_5\theta^5(t)=0\,, \qquad (14)$$

where d_1 is relative damping coefficients, and k_1, k_3 and k_5 are relative restoring coefficients. The homotopy perturbation method is used to *obtain an* approximate analytical solution for the nonlinear *equation* with *initial-boundary conditions***.** *The roll angle can be obtained by*

$$\theta(t)=e^{\frac{-d_1 t}{2}}\left[(l+\alpha)\cos\left(\frac{\gamma t}{2}\right)+\frac{d_1}{\gamma}(l+\alpha)\sin\left(\frac{\gamma t}{2}\right)\right]-\alpha\ , \qquad (15)$$

where α and γ are given in Eqn. (9).The roll velocity and acceleration are, respectively:

$$\frac{d\theta}{dt}=e^{\frac{-d_1 t}{2}}\left[(l+\alpha)\left(\frac{d_1}{\gamma}-\frac{\gamma}{2}\right)\sin\left(\frac{\gamma t}{2}\right)\right], \qquad (16)$$

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$$\frac{d^2\theta}{dt^2} = e^{\frac{-d_1 t}{2}}\left[(l+\alpha)\left(\frac{d_1}{\gamma}-\frac{\gamma}{2}\right)\left(\frac{-d_1}{2}\sin\left(\frac{\gamma t}{2}\right)+\frac{\gamma}{2}\cos\left(\frac{\gamma t}{2}\right)\right)\right]. \qquad (17)$$

The roll angle and velocity for the experimental value of the parameters $d_1 = 0.0126513,\ k_1 = 0.67199703,\ k_3 = -0.5392039,\ k_5 = -0.0867972,\ a = 0.1,\ \omega = 0.1,\ l = 0.3$ are obtained as follows:

$$\theta(t) = e^{-0.0063t}\left[0.2780\cos(0.8062t) + 0.0021\sin(0.8062t)\right] + 0.022. \qquad (18)$$

$$\frac{d\theta}{dt} = e^{-0.0063t}\left[-0.00001\cos(0.8062t) - 0.2247\sin(0.8062t)\right] \qquad (19)$$

Following equations (10) and (11), we can also obtain a moment of restore and moment of damping.

4. NUMERICAL SIMULATION

The analytical solutions obtained by the proposed HPM method for equations (3) and (14) with boundary conditions (4) and (5) are in satisfactory agreement with the numerical solution obtained using Matlab software as shown in Figure 2.

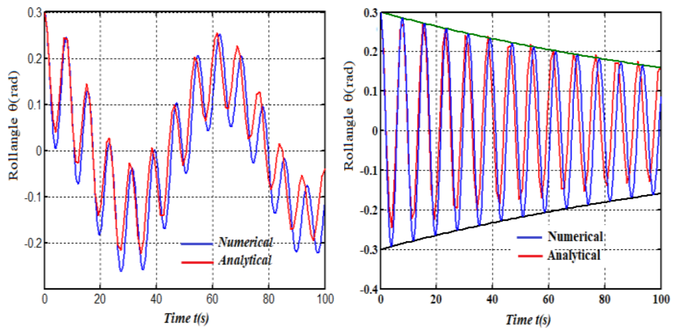

Fig. 2: Comparison of analytical and numerical solutions of roll angle for the experimental value

$d_1 = 0.0126513,\ k_1 = 0.67199703,\ k_3 = -0.5392039,\ k_5 = -0.0867972,\ a = 0.1,\ \omega = 0.1,\ l = 0.3$

Left: Linear damping with wave exciting moment (Eqn. (6)). Right: Linear damping without wave exciting moment (Eqn. (15))

5. DISCUSSION

Eqns. (6) and (15), represent the analytical expressions of the roll angle with and without wave exciting moments, respectively. Both the analytical and numerical results for roll angle for some experimental values of the parameter are presented in fig.2

It shows the roll decay of a ship motion from an initial angle of a radian. It also shows the main properties of damped roll motion. A measure for roll damping can be obtained from the reduction of each successive roll angle maximum [18]. It is also used to find the roll decay test of a naval vessel. If the roll amplitudes fall under a threshold of one degree, the time series is truncated because the rolling period continues to drift away at a rolling motion close to its disappearance. At decay tests with forwarding speed, the roll motion does not entirely, but a slight oscillation of the roll angle, driven by the forward momentum, remains [1].

Equations (6) & (7) represents velocity and Equation (12) is a moment of regeneration, and Equation (13) describes a moment of damping. Figure 3 shows the velocity versus time for some practical values of parameters. From figure 3, it is inferred that the amplitude velocity for the linear damping without exciting wave moment is higher than that of the linear damping with exciting wave moment.

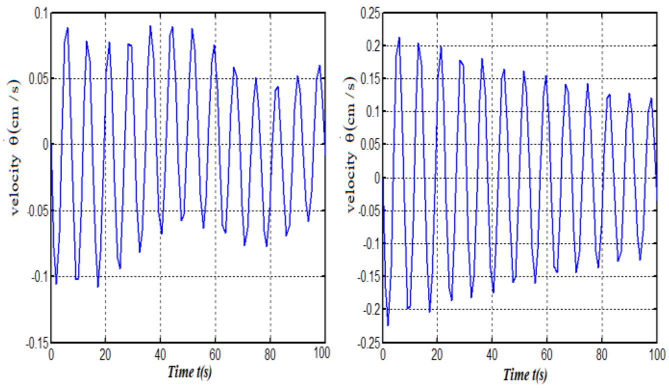

Fig.3 Velocity versus time. Left: Linear damping with wave exciting moment (Eqn. (7)). Right:

Linear damping without wave exciting moment (Eqn. (14)).

Figures 4 represent acceleration versus time. From the figure, it is inferred that amplitude decreases when time increases.

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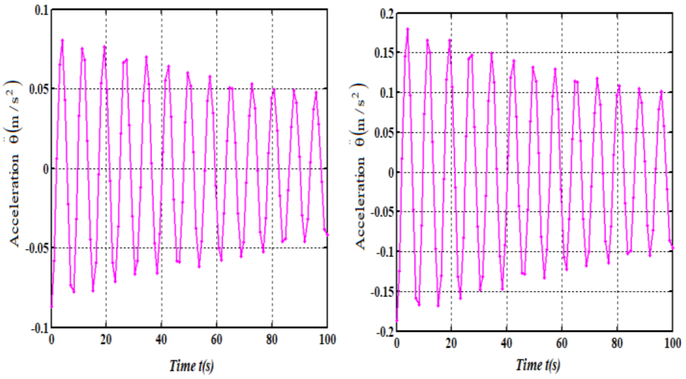

Fig.4 Acceleration versus time. Left: Linear damping with wave exciting moment(Eqn. (8)).

Right: Linear damping without wave exciting moment (Eqn. (15)).

Figures 5 and 6 represent the resorting moment and damping moment for various values of parameters. It is used to measure the bilge keel height. From the figure, it is noted that the period of oscillation is not constant and is slowly increasing in time. Using the envelope function $f=\pm\theta(t=0)\,e^{-d_1 t/2}$ the damping coefficient can be obtained.

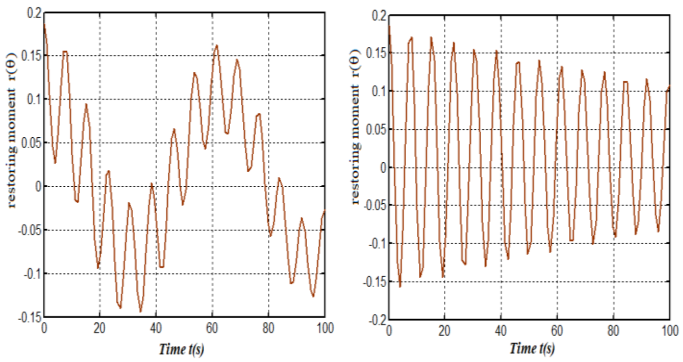

Fig.5 Restoring moment versus time Eqn. (12). Left: Linear damping with wave exciting moment. Right: Linear damping without wave exciting moment.

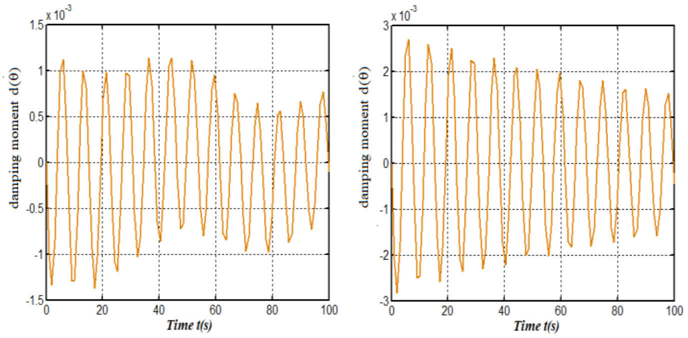

Fig. 6 Damping moment versus time using Eqn. (13). Left: Linear damping with wave exciting moment. Right: Linear damping without wave exciting moment.

The ship's energy loss is due to impacts of friction and gravitational pressure.Using the roll velocity and restoreing moment the ship's energy can be calculated using the below equation.

$$E_{tot}=E_{kin}+E_{pot}=\frac{1}{2}I\theta^2+\int_0^{\theta}\Delta\,\overline{GM}\;\theta\,d\theta \qquad (18)$$

where E_{kin} is the kinetic energy and E_{pot} is the potential energy. The kinetic energy is depends on angular velocity where as potential energy is depends upon inclination of the ships against the restoring moment.

5. CONCLUSIONS

In this paper, a theoretical model describing the nonlinear roll motion of a ship has been discussed. The model represented by a nonlinear time-dependent differential equation has been solved analytically. Approximate analytical expressions of the roll angle, velocity, acceleration, resorting moment and damping moment for all the values of parameters are obtained using the new approach of homotopy perturbation method. The accuracy of the analytical solutions was shown to be very satisfactory when compared with the numerical solution. These analytical results are useful to validate the experimental findings and to find the roll decay test. The analytical model will help to improve the level of design and safety of a ship.

Appendix- A: Approximate analytical solution of nonlinear Eqn. (3) using HPM for linear damping

In this Appendix, we derive the general solution of the nonlinear reaction equation (3) using the new approach homotopy perturbation method. We begin by constructing the homotopy for Eqn. (6) as follows:

$$(1-p)\left[\ddot{\theta}(t)+d_1\dot{\theta}(t)+k_1\theta(t)\right]+p\left[\ddot{\theta}(t)+d_1\dot{\theta}(t)+k_1\theta(t)+k_3\theta^3+k_5\theta^5-a\cos\omega t\right]=0 \qquad (A1)$$

The approximate solution of the Eq. (A1) is

$$\theta=\theta_0+p\theta_1+p^2\theta_2+... \qquad (A2)$$

By equating like powers of p, we get the following linear equations:

$$p^0:\ddot{\theta}_0(t)+d_1\dot{\theta}_0(t)+k_1\theta_0(t)=0 \qquad (A3)$$

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$$p^1: \ddot{\theta}_1(t)+d_1\dot{\theta}_1(t)+k_1\theta_1(t)+k_3\theta_0^3(t=0)+k_5\theta_0^5(t=0)-a\cos\omega t=0$$
(A4)

with boundary conditions for Eqn. (A3) and Eqn. (A4) respectively given by

$$\theta_0(0)=l, \dot{\theta}_0(0)=0 \quad \text{(A5)}$$

$$\theta_1(0)=0, \dot{\theta}_1(0)=0 \quad \text{(A6)}$$

Solving Eqns. (A3) and (A4) with boundary conditions (A5) and (A6) we get:

$$\theta_0(t)=e^{\frac{-d_1 t}{2}}\left[l\cos\left(\frac{\gamma t}{2}\right)+\frac{l d_1}{\gamma}\sin\left(\frac{\gamma t}{2}\right)\right]$$
(A7)

$$\theta_1(t)=e^{\frac{-d_1 t}{2}}\left[\left(\alpha-\beta\left(k_1-\omega^2\right)\right)\cos\left(\frac{\gamma t}{2}\right)+\left(\frac{d_1}{\gamma}\left(\alpha-\beta\left(k_1+\omega^2\right)\right)\sin\left(\frac{\gamma}{2}\right)t\right)\right]$$
$$-\alpha+\beta\left(d_1\omega\sin\omega t+\left(k_1-\omega^2\right)\cos\omega t\right)$$
(A8)

The sum of (A7) and (A8) gives the Eqn. (6).

Appendix- B. Matlab Program for the Numerical Solution of Eqn.(3)

```
function bvp_ode_matlab
options=bvpset('stats','off','RelTol',1e-6);
solinit=bvpinit(linspace(0,100),[0,0]);
sol=bvp4c(@OdeBVP, @OdeBC, solinit, options);
plot(sol.x,sol.y(1,:));
end
function f=OdeBVP(x,y)
%k1=1,k3=1,A=0.2,d1=2
f=[y(2)
%-0.2*y(2)-0.1*y(1)+0.3*(y(1))^3
-0.0476542*y(2)-0.65*y(1)+1.2*y(1).^3-0.105*y(1).^5+0.1*cos(0.1.*x)
];
end
function res=OdeBC(ya,yb)
res=[ya(1)-0.3
ya(2)
];
end
```

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